ELECTRICAL

The X-Transform And Its Use In Sampled Data System Analysis

PREPARED BY
SERVOMECHANISMS LABORATORY
AUBURN UNIVERSITY
C. L. PHILLIPS, TECHNICAL DIRECTOR

SECOND TECHNICAL REPORT
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GEORGE C. MARSHALL SPACE FLIGHT CENTER
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HUNTSVILLE, ALABAMA 35812

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TABLE OF CONTENTS

LIST OF SYMBOLS

G	General transfer function
G_R	Barred open-loop transfer function
Q	Zero-order hold transfer function
T	Sampling period, seconds
U	System input
C	System output
ω_s	Sampling frequency, radians per second
$\triangleq$	Equal by definition
$*$	Sampled signal
$\_\_\_$	Signal from zero-order hold

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FOREWORD

This report is a detailed explanation of the progress made in the sampled-data area of Contract No. NAS8-11116 granted to Auburn Research Foundation, Auburn, Alabama. The contract was awarded October 21, 1963 by the George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama.

The work reported in this document is the results of an investigation by Willie L. McDaniel, Jr., Auburn University, of a special method of analysis for a certain class of sampled-data systems.

SUMMARY

The development of a transform for analyzing a special class of sampled-data systems is given. The transform, called the x-transform, is applicable to sampled-data systems which have all samplers followed by zero-order hold devices.

PERSONNEL

The following named staff members of Auburn University have actively participated on this project

C. H. Weaver - Head Professor of Electrical Engineering

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I. INTRODUCTION

The field of sampled-data systems has been of increasing importance and interest to engineers and scientists over the past fifteen years. Many text books on digital and sampled-data control systems have been written and used in the classroom and industry. The principal tool for the analysis and synthesis of sampled-data control systems in these books is the z-transform.

The z-transform analysis is based on impulse sampling, which can be only approximated in a real system. In the practical system the "impulse" is actually a very narrow pulse and the normal z-transform analysis gives results that are actually incorrect if the sampler is not followed by a data-hold device. The z-transform analysis gives good results when the practical sampler is followed by a hold device. The zero-order hold is a common example of such a device, and it has been described in the literature.[1, 2*] Since the combination of a sampler and zero-order hold appears in practice as a composite device, it would seem feasible to develop a transform theory about this combination.

Doetsch[3] considered such an approach in a combination of devices termed the pulse-former and impulse-lengthener. The general theme of

*Superscripts refer to references listed in the Reference section.

this approach led to the z-transform approach in an open-loop configuration.

Gardner and Barnes[4] developed a theory around jump functions. This approach could be extended to a sampler-hold analysis; however, it was developed as an aid in the solution of linear difference equations with constant coefficients.

Farmanfarma[5, 6, 7] recognized the need for an exact method of analyzing sampled-data systems with finite pulse widths. The p-transform and its theory were developed, and the method provides an exact analysis of finite pulsed linear systems. The p-transform tables included in the references by Farmanfarma make it possible to compute the output of a finite-pulsed system as a continuous function of time. It has been shown that as the pulse width p approaches the sampling period T, the p-transform approaches the ordinary Laplace transform for continuous systems. Therefore, the p-transform is a special case of the Laplace transform.

Another effort toward the analysis of systems with finite width sampling was provided by Tou[8]. The development of the $\mathcal{T}$-transform was based on the delayed z-transform and should be used in conjunction with the modified z-transform.

The approaches to the analysis of systems through finite pulse duration samplers have provided the necessary tools for accurate analyses of sampled-data systems. There is, however, a large class of systems in which the sampler is followed by a zero-order hold circuit. It is this class of systems that is considered in this dissertation.

The purpose of this investigation is the development of the general theory of a transform, called the x-transform, for a sampler-hold combination in open-loop and closed-loop sampled-data control systems. The stability criteria for such systems are investigated. The application of the x-transform method of analysis to various sampled-data systems is also investigated.

II. THE X-TRANSFORM THEORY

In most feedback control systems employing sampled-data, the high frequency sidebands occurring as the result of the sampling operation must be removed before the signal is applied to the continuous part of the system. This step becomes absolutely necessary since the continuous signal must be reconstructed from the sampled signal. Thus sampled-data systems must possess either inherently or by design some type of hold device or data reconstruction device. The zero-order hold circuit is quite popular and has been thoroughly studied in the literature.[1-2]

Definition of the X-Transform

Since the hold device and the sampler may occur as an entity, it would seem feasible to treat them as such in a transform analysis. Figure 1 shows this combination in block diagram form.

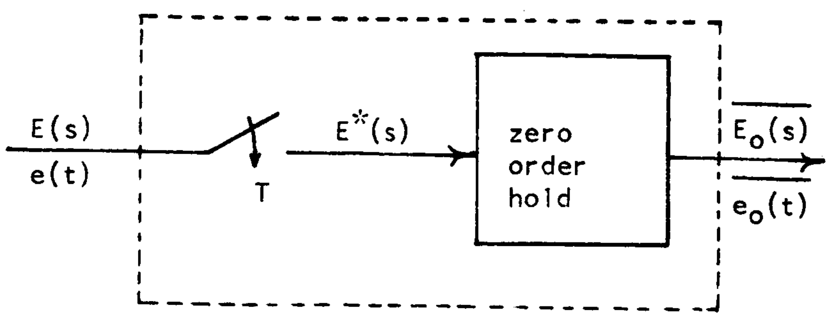


Fig. 1. - Sampler and zero-order hold combination.

It is assumed that the sampler is ideal. The output of the zero-order hold (z.o.h.) device is given by

$$\overline{e_0(t)} = e(0)\,u(t) - e(0)u(t-T) + e(T)u(t-T) - e(T)u(t-2T) + \ldots \qquad \text{(II-1)}$$

Taking the Laplace transform of both sides of equation (II-1) yields

$$\overline{E_0(s)} = \frac{e(0)}{s} - \frac{e(0)e^{-Ts}}{s} + \frac{e(T)e^{-Ts}}{s} - \frac{e(T)e^{-2Ts}}{s} + \ldots \qquad \text{(II-2)}$$

Equation (II-2) may be rearranged to give

$$\overline{E_0(s)} = \frac{1-e^{-Ts}}{s}\left[e(0) + e(T)e^{-Ts} + e(2T)e^{-2Ts} + \ldots\right] \qquad \text{(II-3)}$$

or

$$\overline{E_0(s)} = \sum_{n=0}^{\infty} e(nT)\,\frac{e^{-nTs}}{s} - \sum_{n=0}^{\infty} e(nT)\,\frac{e^{-(n+1)Ts}}{s} \qquad \text{(II-4)}$$

It is convenient to introduce a change in variable by setting

$$x^n \triangleq \frac{e^{-nTs}}{s} \qquad \text{(II-5)}$$

Substituting (II-5) into (II-4) results in

$$E(x) = \sum_{n=0}^{\infty} e(nT)(x^n - x^{n+1}) \qquad \text{(II-6)}$$

It should be observed that $x^0 \neq 1$, but $x^0 = 1/s$. Therefore, (II-6) is

$$E(x) = e(0)(x^0 - x) + e(T)(x - x^2)$$

$$+ e(2T)(x^2 - x^3) + \ldots$$

$$= e(0)\, x^0 + x\left[e(T) - e(0)\right] + \ldots \qquad \text{(II-7)}$$

By definition, the x-transform of e(t) is

$$\mathcal{X}\left[e(t)\right] \triangleq E(x) \triangleq \overline{E(s)}\Bigg|_{\frac{e^{-nTs}}{s} = x^n} \qquad \text{(II-8)}$$

Evaluation of Some X-Transforms

Definitions for x-algebra

In the evaluation of x-transforms some manipulations with x will be necessary. The following definitions are now made and will be retained for the remainder of the x-transform work:

a. $x^n = e^{-nTs}/s$ (II-9)

b. $x_p^m = e^{-mTs}$ (II-10)

Then

$$x^n x_p^m = \frac{e^{-nTs}}{s} e^{-mTs} = x^{m+n} \qquad \text{(II-11)}$$

If x^r is factored from x^n, the remainder is x_p^{n-r} or

$$x^n = x^r x_p^{n-r} \qquad \text{(II-12)}$$

Note that, while $x^0 = 1/s$, $x_p^0 = 1$.

<u>The x-transform of a unit step, u(t)</u>

$$E(x) = e(0)(x^0 - x) + e(T)(x - x^2) + \ldots \qquad \text{(II-13)}$$

$$= e(0)\, x^0 + \left[e(T) - e(0)\right] x + \ldots$$

$$= x^0 + (1 - 1)\, x + \ldots$$

$$E(x) = \mathcal{X}\left[u(t)\right] = x^0 \qquad \text{(II-14)}$$

The x-transform of a ramp, $e(t) = t$

$$E(x) = 0(x^0 - x) + T(x - x^2) + 2T(x^2 - x^3) + \dots \qquad \text{(II-15)}$$

$$= Tx + Tx^2 + Tx^3 + \dots$$

$$= Tx \left[1 + x_p + x_p^2 + x_p^3 + \dots\right]$$

$$E(x) = \frac{Tx}{1 - x_p} \qquad \text{(II-16)}$$

The x-transform of the exponential, $e(t) = e^{-at}$

$$E(x) = x^0 + x(e^{-aT} - 1) + x^2(e^{-2aT} - e^{-aT}) + \dots \qquad \text{(II-17)}$$

$$E(x) = x^0 + (e^{-aT} - 1)\, x \left[\frac{1}{1 - e^{-aT}\, x_p}\right] \qquad \text{(II-18)}$$

$$E(x) = \frac{(x^0 - x)}{1 - e^{-aT}\, x_p} \qquad \text{(II-19)}$$

The x-transform of $\sin \omega t$ and $\cos \omega t$

The x-transform for sinusoids can be obtained from (II-19) by setting $a = -j\omega$. The $\mathcal{X}\left[\cos \omega t\right]$ is the real part of the subsequent equation and the $\mathcal{X}\left[\sin \omega t\right]$ is the imaginary part. The results are

$$\mathcal{X}\left[\cos \omega t\right] = \frac{x^0 - x + (x^2 - x) \cos \omega T}{x_p^2 - 2x_p \cos \omega T + 1} \tag{II-20}$$

and

$$\mathcal{X}\left[\sin \omega t\right] = \frac{x(1 - x_p) \sin \omega T}{x_p^2 - 2x_p \cos \omega T + 1} \tag{II-21}$$

A table of x-transforms is provided in Appendix A, where x and x_p have been made indistinguishable from each other by dropping the subscript p. If in using this table, it becomes desirable to return to the $x - x_p$ form of the transform, x appears only in the numerator and x_p appears only in the denominator. Note that in setting up this table, (II-11) has been used for eliminating x_p in the numerator. If x^n in the numerator is factored, it must be factored according to (II-12); i.e., $x^n = x^r \, x_p^{n-r}$.

Because of the simplicity and usefulness of the power series method for obtaining the inverse transform, this method is generally employed. The expansion of E(x) in a power series will be made void of x_p, thus eliminating the requirement of distinguishing between x and x_p. However, when inverting by the partial fraction expansion or by the inversion integral, where x and x_p both appear, it will be found convenient to distinguish between the two.

The Inverse X-Transform

Power series method

The inverse x-transform may be obtained through the power series method. E(x) is expanded in a power series in powers of x. The coefficient of the term x^n corresponds to the change of the value of the time function $\overline{e(t)}$ at the n-th sampling instant, as can be seen from (II-7). The function $\overline{e(t)}$ is constant between sampling instants.

As an example, consider the x-transform of e(t) = t.

$$\mathcal{X}[t] = \frac{Tx}{1 - x} \tag{II-22}$$

$$E(x) = Tx + Tx^2 + Tx^3 + \ldots \tag{II-23}$$

A plot of the inverse of (II-23) is shown in Figure 2.

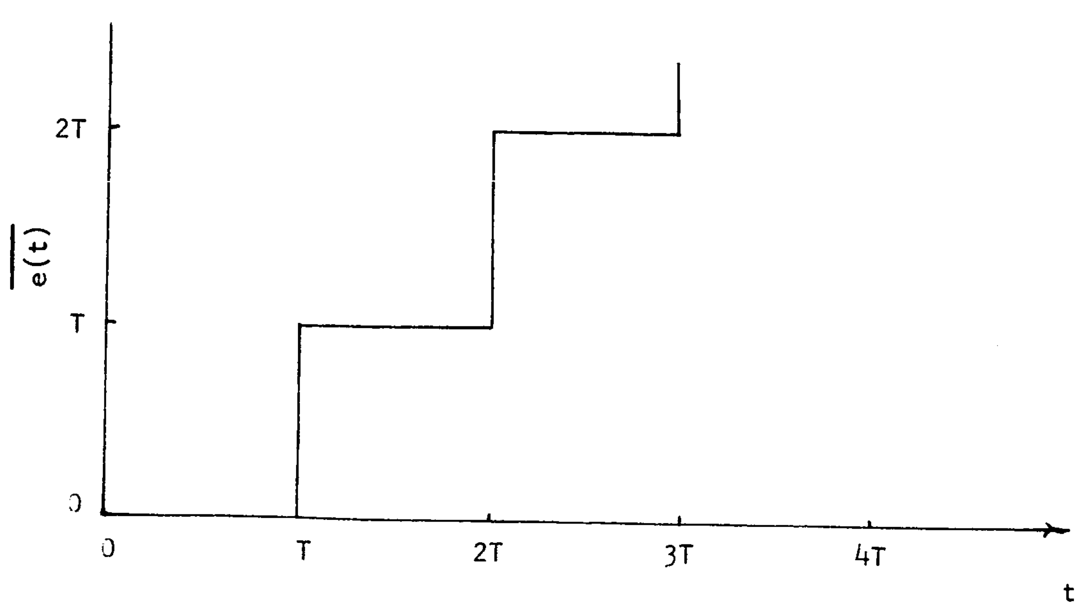


Fig. 2. - Output of a sampled-hold device with a ramp input.

Equation (II-23) can be rearranged as

$$E(x) = T(x - x^2) + 2T(x^2 - x^3) + \ldots \qquad \text{(II-24)}$$

In this form the coefficient of each x-difference term is the output over the corresponding sampling period. Of course the result is the same as that shown in Figure 2.

The partial fraction method

In the analysis of a system having continuous signals, the partial fraction expansion of the Laplace transform of E(s), where E(s) is a rational fraction in s, is given as

$$E(s) = \frac{A}{s + a} + \frac{B}{s + b} + \frac{C}{s + c} + \ldots \qquad \text{(II-25)}$$

from which the inverse Laplace transform may be obtained as

$$e(t) = Ae^{-at} + Be^{-bt} + Ce^{-ct} + \ldots \qquad \text{(II-26)}$$

It would seem that E(x) could be put in a similar form; that is,

$$E(x) = \frac{A}{x - a} + \frac{B}{x - b} + \frac{C}{x - c} + \ldots \qquad \text{(II-27)}$$

However, an investigation of the x-transform table in Appendix A reveals that the terms on the right-hand side of (II-27) do not appear as such. The x-transform of Ae^{-at} is $A(x^0 - x)/\ (1 - e^{-aT}\ x)$.

Therefore, if each term in the partial fraction expansion is expressed in the form of $A(x^0 - x) / (1 - e^{-aT} x)$, the inverse x-transform may be expressed as a sum of exponential functions. It is desirable then to expand $E(x) / (x^0 - x)$ in the form of (II-27) and then to multiply each of the expanded terms by $(x^0 - x)$. An example of the use of the partial fraction expansion method for finding the inverse x-transform is given in Example 1 in Appendix B.

<u>The inversion formula method</u>

The time function $\overline{e(t)}$ may be obtained from $E(x)$ by an inversion formula, which is based on the real inversion formula of the Laplace transform. The derivation of the inversion formula is as follows:

The inverse transform in s is

$$e(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} E(s)\, e^{ts}\, ds \qquad \text{(II-28)}$$

The value of e(t) at the n-th sampling instant is

$$e(nT) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} E(s)\, e^{nTs}\, ds \qquad \text{(II-29)}$$

The constant c is greater than σ_a where σ_a is the abscissa of absolute convergence of the Laplace transform. The path for the integration in the s-plane is shown in Fig. 3a.

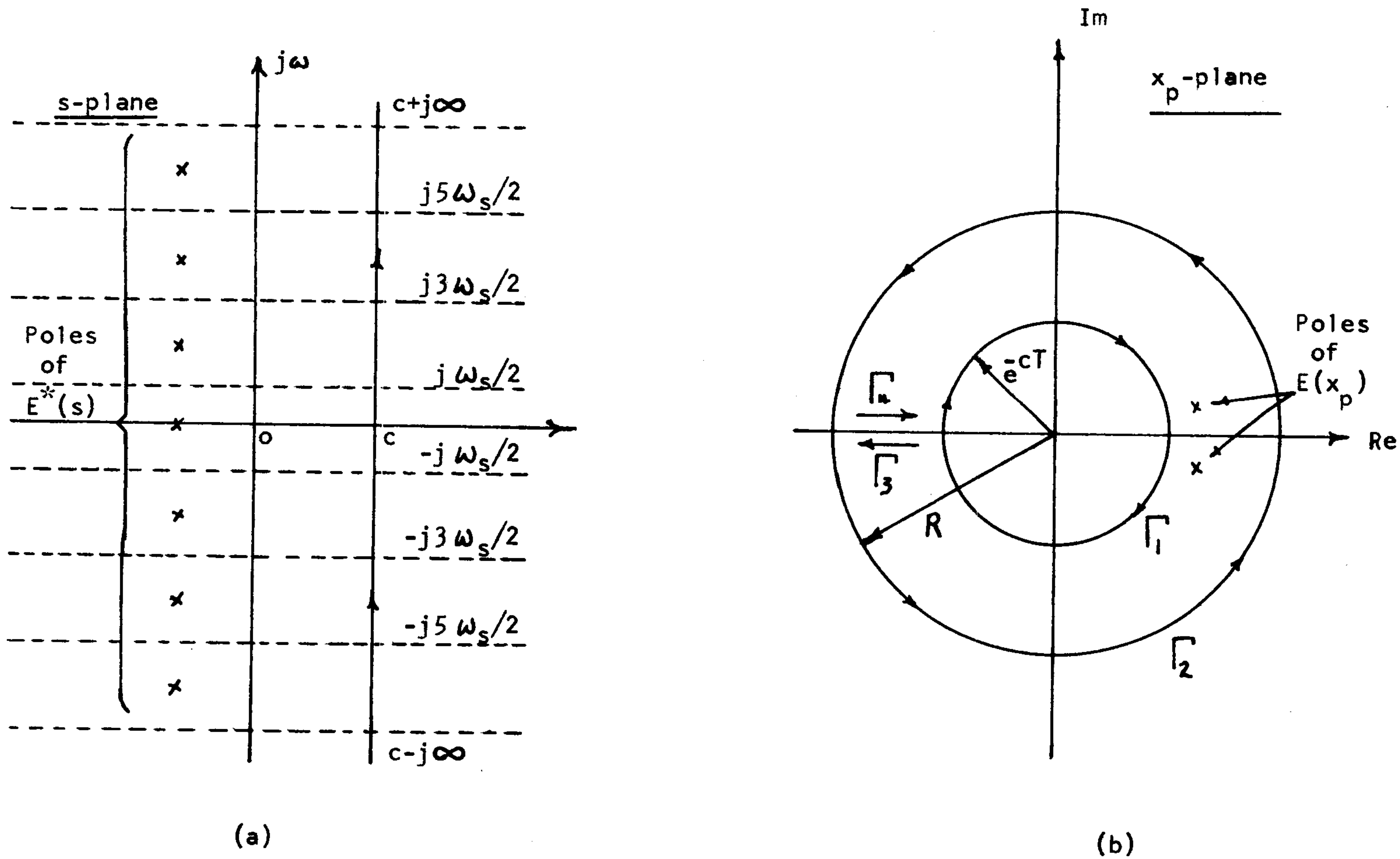


Fig. 3. - Path of integration of the inversion formula in the (a) s-plane (b) the x_p-plane.

The corresponding path in the x_p-plane is a circle which encloses all of the singularities of $E(x)x_p^{-(n+1)} / (x^0 - x)$. x_p is a pseudo x-transform as defined in (II-10). It is noted that

$$x_p \triangleq e^{-Ts} = z^{-1} \tag{II-30}$$

where z is the variable of the ordinary z-transform. A detailed explanation of x_p is given in the introductory portion of Chapter IV. It is noticed that the path of integration in the s-plane passes through the periodic strips vertically; therefore, the integration in (II-29) may be broken up into a sum of integrals given by

$$e(nT) = \frac{1}{2\pi j} \sum_{k=-\infty}^{\infty} \int_{c + (k - \frac{1}{2})j\omega_s}^{c + (k + \frac{1}{2})j\omega_s} E(s)e^{nTs}ds \tag{II-31}$$

where $\omega_s = 2\pi/T$. Replacing s by $s + jk\omega_s$ alters (II-31) to

$$e(nT) = \frac{1}{2\pi j} \sum_{k=-\infty}^{\infty} \int_{c-j\omega_s/2}^{c+j\omega_s/2} E(s+jk\omega_s) \cdot$$

$$e^{nT(s+jk\omega_s)} \; d(s + jk\omega_s) \tag{II-32}$$

Interchanging the summation and integration signs and simplifying (II-32) gives

$$e(nT) = \frac{1}{2\pi j} \int_{c-j\omega_s/2}^{c+j\omega_s/2} \sum_{k=-\infty}^{\infty} E(s+jk\omega_s)\, e^{nTs}\, ds \qquad \text{(II-33)}$$

However, in normal impulse sampling

$$E^*(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} E(s + jk\omega_s) \; ; \; \left[e(0) = 0 \right] \qquad \text{(II-34)}$$

and (II-33) may be written as

$$e(nT) = \frac{T}{2\pi j} \int_{c-j\omega_s/2}^{c+j\omega_s/2} E^*(s)\, e^{nTs}\, ds \qquad \text{(II-35)}$$

The transfer function of a zero-order hold is

$$Q(s) = \frac{1 - e^{-Ts}}{s} \qquad \text{(II-36)}$$

Therefore, from Figure 1 it is seen that

$$E^*(s) = \frac{\overline{E}(s)}{Q(s)} = \frac{\overline{E}(s)}{\dfrac{1 - e^{-Ts}}{s}} \qquad \text{(II-37)}$$

Introducing (II-37) into (II-35) and noting that the presence of the zero-order hold causes the output to be constant between sampling instants, one obtains

$$e(nT) \left[u(nT) - u\left[(n-1)T\right]\right]$$

$$= \frac{T}{2\pi j} \int_{c - j\omega_s/2}^{c + j\omega_s/2} \frac{\overline{E}(s)\, e^{nTs}}{\dfrac{1 - e^{-Ts}}{s}} \, ds \qquad \text{(II-38)}$$

In the evaluation of (II-38), a change in variable is found to be convenient. Let

$$x_p \triangleq e^{-Ts} \qquad \text{(II-39)}$$

Then

$$dx_p = -Te^{-Ts} \qquad \text{(II-40)}$$

or

$$ds = \frac{dx_p}{-Tx_p} \qquad \text{(II-41)}$$

Substituting (II-39) and (II-41) into (II-38) gives

$$e(nT)\left[u(nT) - u\left[(n-1)T\right]\right]$$

$$= \frac{T}{2\pi j}\oint_{\Gamma_1} \frac{E(x)}{-(x^n - x^{n+1})Tx_p}\, dx_p \qquad \text{(II-42)}$$

It should be noted that the integrand of (II-42) is a function only of x_p. The path of integration, Γ_1, is the circle obtained by mapping the line $s = c + j\omega$ from the s-plane onto the x_p-plane as shown in Figure 3b. The area to the left of c is mapped outside the circle and the area to the right of c is mapped inside the circle. Simplifying (II-42) gives

$$e(nT)\left[u(nT) - u\left[(n-1)T\right]\right]$$

$$= -\frac{1}{2\pi j}\oint_{\Gamma_1} \frac{E(x)\, x_p^{-(n+1)}}{x^0 - x}\, dx_p \qquad \text{(II-43)}$$

$$= -\sum \text{Residues of } E(x)\, x_p^{-(n+1)}/(x^0 - x)$$

$$\text{at the poles of } x_p^{-(n+1)} \qquad \text{(II-44)}$$

The residues in (II-44) may be obtained by the following relationship:

$$\operatorname{Res}\left\{\frac{E(x_p)}{x_p^{(n+1)}}\;,\;0\right\}$$

$$= \frac{1}{n!}\lim_{x_p \to 0}\frac{d^n}{dx_p^n}\left\{x_p^{n+1}\left[\frac{E(x_p)}{x_p^{n+1}}\right]\right\} \qquad \text{(II-45)}$$

It is also seen that, if $\left[E(x) / (x^0 - x)\right] x_p^{-(n+1)}$ is expressed in a series, such as $\left[\sum_{n=0}^{\infty} K_n x_p^n\right] \Big/ x_p^{n+1}$, the coefficients of the x^{-1} term are the residues K_n, which for a given n is the value of e(t) over the nth sampling instant as was indicated in (II-24).

The integral of (II-43) may also be evaluated through the aid of the contours shown in Figure 3b. Thus,

$$-\oint_{\Gamma_1}\frac{E(x_p)}{x_p^{n+1}}\,dx_p + \int_{\Gamma_3}\frac{E(x_p)}{x_p^{n+1}}\,dx_p + \oint_{\Gamma_2}\frac{E(x_p)}{x_p^{n+1}}\,dx_p$$

$$+\int_{\Gamma_4}\frac{E(x_p)}{x_p^{n+1}}\,dx_p$$

$$= 2\pi j \sum \text{Residues of } E(x_p)/x_p^{n+1}$$

$$\text{at poles of } E(x_p) \qquad \text{(II-46)}$$

It may be observed that the integrals along the paths Γ_3 and Γ_4 cancel each other. The integral along Γ_2, which is chosen as a circle with center at the origin, will be evaluated with R approaching infinity, thus assuring that all the poles of $E(x_p)$ are in the desired region. Therefore, with a change of variable the integral along Γ_2 becomes

$$\int_0^{2\pi} \frac{E(Re^{j\theta})j\,d\theta}{R^n\,e^{j\theta n}} = 0 \qquad \text{(II-47)}$$

if $E(Re^{j\theta})$ approaches zero as R approaches infinity. Stipulating this condition on the integral along Γ_2, (II-46) becomes

$$\oint_{\Gamma} \frac{E(x_p)}{x_p^{n+1}}\,dx_p = -\sum \text{Residues of } E(x_p)\,/\,x_p^{n+1}$$

$$\text{at poles of } E(x_p) \qquad \text{(II-48)}$$

The inverse x-transform may be obtained by (II-44) or (II-48).

As an example, determine the inverse x-transform of

$$E(x) = \frac{x^0 - x}{1 - e^{-aT}\,x_p} \qquad \text{(II-49)}$$

Using (II-48), one obtains

$$e(nT)\left[u(nT) - u\left[(n-1)T\right]\right]$$

$$= -\frac{1}{2\pi j}\oint \frac{(x^0 - x)\, x_p^{-(n+1)}}{(x^0 - x)(1 - e^{-aT}x_p)}\, dx_p \qquad \text{(II-50)}$$

$$= -\sum \text{Residues of } \frac{x_p^{-(n+1)} e^{aT}}{-(x_p - e^{aT})} \text{ at } x_p = e^{aT}$$

$$= (e^{aT})^{-(n+1)}\, e^{aT}$$

$$= e^{-anT} \qquad \text{(II-51)}$$

The value of $\overline{e(t)}$ is obtained from (II-51) as

$$\overline{e(t)} = \sum_{n=0}^{\infty} e^{-anT}\left[u(t - nT) - u\left[t - (n+1)\,T\right]\right] \qquad \text{(II-52)}$$

For a second example of the use of the inversion formula see Example 2 in Appendix B.

Theorems of the X-Transform

Addition and subtraction

If $e_1(t)$ and $e_2(t)$ are Laplace transformable and

$$E_1(x) = \mathcal{X}\left[e_1(t)\right] , \; E_2(x) = \mathcal{X}\left[e_2(t)\right] \quad \text{then}$$

$$\mathcal{X}\left[e_1(t) \pm e_2(t)\right] = E_1(x) + E_2(x) \tag{II-53}$$

Proof: By definition

$$\mathcal{X}\left[e_1(t) \pm e_2(t)\right] = \sum_{n=0}^{\infty} \left[e_1(nt) \pm e_2(nt)\right] (x^n - x^{n+1}) \tag{II-54}$$

$$\mathcal{X}\left[e_1(t) \pm e_2(t)\right] = \sum_{n=0}^{\infty} e_1(nt)(x^n - x^{n+1})$$

$$\pm \sum_{n=0}^{\infty} e_2(nt)(x^n - x^{n+1}) \tag{II-55}$$

$$= E_1(x) \pm E_2(x) \quad \text{Q.E.D.} \tag{II-56}$$

Multiplication by a constant

If $E(x)$ is the x-transform of $e(t)$, then

$$\mathcal{X}\left[ae(t)\right] = a\,\mathcal{X}\left[e(t)\right] = aE(x) \tag{II-57}$$

Proof: By definition

$$\mathcal{X}\left[ae(t)\right] = \sum_{n=0}^{\infty} ae(nt)(x^n - x^{n+1}) \qquad \text{(II-58)}$$

$$= a \sum_{n=0}^{\infty} e(nt)(x^n - x^{n+1}) \qquad \text{(II-59)}$$

$$= aE(x) \quad \text{Q.E.D.} \qquad \text{(II-60)}$$

<u>Shifting theorem</u>

If $\mathcal{X}\left[e(t)\right] = E(x)$, then

$$\mathcal{X}\left[e(t + T)\right] = x^{-1}\left[E(x) - e(0)(x^0 - x)\right] \qquad \text{(II-61)}$$

<u>Proof</u>: By definition

$$\mathcal{X}\left[e(t + T)\right] = \sum_{n=0}^{\infty} e\left[(n + 1)\, T\right] (x^n - x^{n+1}) \qquad \text{(II-62)}$$

$$= x_p^{-1} \sum_{n=0}^{\infty} e\left[(n + 1)T\right] (x^n - x^{n+1})x_p$$

$$= x_p^{-1} \sum_{n=0}^{\infty} e\left[(n + 1)T\right] (x^{n+1} - x^{n+2}) \qquad \text{(II-63)}$$

Let $k = n + 1$. Therefore,

$$\mathcal{X}\left[e(t + T)\right] = x_p^{-1} \sum_{k=1}^{\infty} e(kT)(x^k - x^{k+1}) \qquad \text{(II-64)}$$

Now in order to make the summation over k from zero to infinity, it is necessary to add and subtract $e(0)(x^0 - x)$. Thus

$$\mathcal{X}\left[e(t + T)\right] = x_p^{-1} \left[\sum_{k=0}^{\infty} e(kT)(x^k - x^{k+1}) \right.$$

$$\left. - e(0)(x^0 - x) \right] \qquad \text{(II-65)}$$

$$= x^{-1} \left[E(x) - e(0)(x^0 - x) \right] \quad \text{Q.E.D.} \qquad \text{(II-66)}$$

Since (II-66) is in a product form, the subscript p is omitted, but understood.

<u>Corollary I.</u> If $\mathcal{X}\left[e(t)\right] = E(x)$, then

$$\mathcal{X}\left[e(t + 2T)\right] = x^{-2} \left[E(x) - e(0)(x^0 - x)\right]$$

$$- x^{-1} e(T)(x^0 - x) \qquad \text{(II-67)}$$

<u>Proof</u>: Using the Shifting Theorem results in

$$\mathcal{Z}\left[e(t + 2T)\right] = \mathcal{Z}\left[e(\overleftrightarrow{t + T} + T)\right]$$

$$= x^{-1}\left[\mathcal{Z}\left[e(t + T)\right] - e(t + T)\cdot(x^{0} - x)\Big|_{t = 0}\right] \qquad \text{(II-68)}$$

Reapplying equation (II-61),

$$\mathcal{Z}\left[e(t + 2T)\right] = x^{-2}\left[E(x) - e(0)(x^{0} - x)\right]$$

$$- x^{-1}\, e(T)(x^{0} - x) \qquad \text{Q.E.D.} \qquad \text{(II-69)}$$

The extension of Corollary I to $\mathcal{Z}\left[e(t + mT)\right]$ can be made resulting in

$$\mathcal{Z}\left[e(t + mT)\right] = x^{-m}\left[E(x) - \sum_{n=0}^{m-1} e(nT)(x^{0} - x)x^{n}\right] \qquad \text{(II-70)}$$

<u>Corollary II.</u> If $\mathcal{Z}\left[e(t)\right] = E(x)$, then

$$\mathcal{Z}\left[e(t - nT)\, u(t - nT)\right] = x^{n}\, E(x) \qquad \text{(II-71)}$$

Proof: By definition

$$\mathcal{X}\left[e(t - nT)\ u(t - nT)\right] = \sum_{m=0}^{\infty} e\left[(m - n)T\right]\ u\left[(m - n)T\right] \cdot (x^{m} - x^{m+1}) \tag{II-72}$$

$$= x^{n} \sum_{m=0}^{\infty} e\left[(m - n)T\right]\ u\left[(m - n)T\right] \cdot (x^{m-n} - x^{m-n+1}) \tag{II-73}$$

Let $k = m - n$. Therefore,

$$\mathcal{X}\left[e(t - nT)\ u(t - nT)\right] = x^{n} \sum_{k=-n}^{\infty} e(kT)\ u(kT) \cdot (x^{k} - x^{k+1}) \tag{II-74}$$

However, $e(kT)\ u(kT) = 0$ for $k < 0$. Therefore,

$$\mathcal{X}\left[e(t - nT)\ u(t - nT)\right] = x^{n} \sum_{k=0}^{\infty} e(kT)\ u(kT)(x^{k}-x^{k+1}) \tag{II-75}$$

$$= x^{n}\ E(x) \qquad \text{Q.E.D.} \tag{II-76}$$

Initial value theorem

If $\mathcal{Z}\left[e(t)\right] = E(x)$ and if $\lim_{x \to 0} \left[\dfrac{E(x)}{x^0 - x} \right]_{x^0 = 1}$ exists, then

$$\lim_{t \to 0} \overline{e(t)} = \lim_{x \to 0} \left[\frac{E(x)}{x^0 - x} \right]_{x^0 = 1} \tag{II-77}$$

Proof: By definition

$$\frac{E(x)}{x^0 - x} = \frac{1}{x^0 - x} \sum_{n=0}^{\infty} e(nT)(x^n - x^{n+1}) \tag{II-78}$$

$$\frac{E(x)}{x^0 - x} = e(0) + \frac{e(T)(x - x^2)}{(x^0 - x)} + \frac{e(2T)(x^2 - x^3)}{(x^0 - x)} + \dots \tag{II-79}$$

Taking the limit as x approaches zero of both sides of (II-79) and setting x^0 equal to one, results in

$$\lim_{t \to 0} \overline{e(t)} = e(0) = \lim_{x \to 0} \left[\frac{E(x)}{x^0 - x} \right]_{x^0 = 1} \qquad \text{Q.E.D.} \tag{II-80}$$

It may be seen from (II-9) and (II-10) that x^n approaches zero as T approaches infinity for $n \neq 0$, and x_p^m approaches zero as T approaches infinity; therefore, no distinction is made in x and x_p. x^0 is independent of T. This can also be seen from the relationship $x = x^0 x_p$. The limit as x approaches zero with $x^0 = 1$ also assures that x_p approaches zero.

Final value theorem

If $\mathcal{X}\left[e(t)\right] = E(x)$, then

$$\lim_{t\to\infty} \overline{e(t)} = \lim_{x\to 1} E(x)\Big|_{x^0 = 1} \qquad \text{(II-81)}$$

Proof: From equation (II-65) and the subtraction theorem, one obtains

$$\mathcal{X}\left[e(t + T) - e(t)\right] = x_p^{-1}\left[E(x) - e(0)(x^0 - x)\right] - E(x) \qquad \text{(II-82)}$$

Note that the subscript p has been included here, but it will be shown later that its inclusion is unnecessary in the application of the theorem. Rearranging (II-82) gives

$$\mathcal{X}\left[e(t + T) - e(t)\right] = E(x)\left[x_p^{-1} - 1\right]$$

$$- e(0)\, x_p^{-1}\,(x^0 - x) \qquad \text{(II-83)}$$

However, by definition

$$\mathcal{Z}\left[e(t + T) - e(t)\right] = \lim_{k\to\infty} \sum_{n=0}^{k} \left[e(n+1)T - e(nT)\right] (x^n - x^{n+1}) \qquad \text{(II-84)}$$

Expanding (II-84) gives

$$\mathcal{Z}\left[e(t + T) - e(t)\right] = \lim_{k\to\infty} \Big[- e(0)\ x^0\ (1 - x_p) + e(T)(x^0 - x)(1 - x_p) + e(2T)(x - x^2)(1 - x_p) + \ldots + e(kT)(x^{k-1} - x^k)(1 - x_p) + e\left[(k + 1)T\right]\ x^k\ (1 - x_p)\Big] \qquad \text{(II-85)}$$

and

$$\frac{\mathcal{Z}\left[e(t + T) - e(t)\right]}{1 - x_p} = \lim_{k\to\infty} \Big[- e(0)\ x^0 + e(T)(x^0 - x) + e(2T)(x - x^2) + \ldots + e(kT)(x^{k-1} - x^k) + e\left[(k+1)T\right]\ x^k\Big] \qquad \text{(II-86)}$$

It is observed that, if the limit is taken of each side of (II-86) as x approaches one with x^0 equal to one, (II-86) becomes

$$\lim_{x\to 1} \left. \frac{\mathcal{X}\left[e(t+T) - e(t)\right]}{1 - x_p} \right|_{x^0 = 1} = e(\infty) - e(0) \tag{II-87}$$

However, from (II-83) one obtains

$$\frac{\mathcal{X}\left[e(t+T) - e(t)\right]}{1 - x_p} = \frac{E(x)}{x_p} - \frac{e(0)\, x^0}{x_p} \tag{II-88}$$

Taking the limit of each side of (II-88) as x approaches one with x^0 equal to one gives

$$\lim_{x\to 1} \left. \frac{\mathcal{X}\left[e(t+T) - e(t)\right]}{1 - x_p} \right|_{x^0 = 1} = \lim_{x\to 1} \left. E(x) \right|_{x^0=1} - e(0) \tag{II-89}$$

It should be observed that $x = x^0\, x_p$. Therefore, when x approaches one with x^0 equal to one, x_p also approaches one. Equating the right-hand sides of (II-87) and (II-89) gives

$$e(\infty) - e(0) = \lim_{x\to 1} \left. E(x) \right|_{x^0 = 1} - e(0) \tag{II-90}$$

or

$$e(\infty) = \lim_{x \to 1} E(x) \Big|_{x^0 = 1} \qquad \text{Q.E.D.} \qquad \text{(II-91)}$$

An example of the application of the final value theorem is given in Example 6, Appendix B.

Complex translation

If the $\mathcal{X}\left[e(t)\right]$ is E(x), then

$$\mathcal{X}\left[e^{\pm at}e(t)\right] = (1 - x)\, E'\,(xe^{\pm aT}) \qquad \text{(II-92)}$$

where $E'(x)$ is $\dfrac{E(x)}{1 - x}$

Proof: By definition

$$\mathcal{X}\left[e^{\pm at}\, e(t)\right] = \sum_{n=0}^{\infty} e(nT)\, e^{\pm anT}\, (x^n - x^{n+1}) \qquad \text{(II-93)}$$

$$= (1 - x_p) \sum_{n=0}^{\infty} e(nT)(e^{\pm aT}\, x)^n \qquad \text{(II-94)}$$

Let $xe^{\pm aT} = x_1$, then

$$\mathcal{X}\left[e^{\pm at}\, e(t)\right] = (1 - x_p) \sum_{n=0}^{\infty} e(nT)\, x_1^n \tag{II-95}$$

$$= (1 - x_p)\, E'(x_1) \tag{II-96}$$

$$= (1 - x)\, E'\,(xe^{\pm aT}) \qquad \text{Q.E.D.} \tag{II-97}$$

Examples utilizing the x-transform theorems have been included in Appendix B.

Relationship Between X-Transform and Z-Transform

The definition of the x-transform has been given by equation (II-5). The definition of the z-transform is given in the literature as

$$z \triangleq e^{sT} \tag{II-98}$$

The relationship between the x-transform and the z-transform will now be derived.

By definition

$$\overline{E(s)} = \sum_{n=0}^{\infty} e(nT) \left[\frac{e^{-nTs} - e^{-(n+1)Ts}}{s} \right] \tag{II-99}$$

$$= \sum_{n=0}^{\infty} e(nT)\, e^{-nTs} \left[\frac{1 - e^{-Ts}}{s} \right] \tag{II-100}$$

$$\frac{\overline{E(s)}}{\frac{1 - e^{-Ts}}{s}} = \sum_{n=0}^{\infty} e(nT)\, e^{-nTs} \qquad \text{(II-101)}$$

Therefore,

$$\frac{\mathcal{L}\left[e(t)\right]}{x^0 - x} = \mathcal{Z}\left[e(t)\right]_{z^{-1} = x} \qquad \text{(II-102)}$$

III. THE TRANSFER FUNCTION

The transfer function for a continuous system is $G(s) = C(s)/E(s)$ for the open-loop case and $G(s) \,/\, \left[1 + G(s)\, H(s)\right]$ for the closed-loop case. The transfer function for a sampled-data-hold system in the open-loop and closed-loop configuration will now be derived from the basic block diagrams for each of the above cases.

Open-Loop

The derivation of the open-loop transfer function depends on the block diagram[3] representation of Figure 4. In the x-transform analysis,

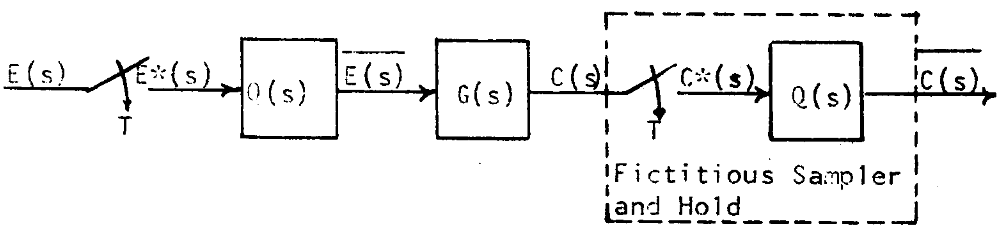


Fig. 4. - Open-loop sampled-data system.

the ideal sampler is always followed by a zero-order hold device. In Figure 4 the zero-order hold is designated by $Q(s)$. $G(s)$ is the plant and the continuous output is $C(s)$. From the block diagram of Figure 4, the following equations can be written or defined:

$$Q(s) \triangleq \frac{1 - e^{-Ts}}{s} \tag{III-1}$$

$$R(s) \triangleq G(s)\, Q(s) \qquad \text{(III-2)}$$

$$G_R(s) = \mathcal{L}\left[g_R(t)\right] \triangleq G(s) \,/\, s \qquad \text{(III-3)}$$

Then

$$r(t) = \mathcal{L}^{-1}\left[R(s)\right] = g_R(t)\, u(t) - g_R(t - T)u(t-T) \qquad \text{(III-4)}$$

Now,

$$C(s) = R(s)\, E^*(s) \qquad \text{(III-5)}$$

Starring (III-5) gives

$$C^*(s) = R^*(s)\, E^*(s) \qquad \text{(III-6)}$$

where the starred transform indicates normal impulse sampling. Then

$$\overline{C(s)} = Q(s)\, C^*(s)$$

$$= Q(s)\, E^*(s)\, R^*(s) \qquad \text{(III-7)}$$

Therefore,

$$\overline{C(s)} = \overline{E(s)}\, R^*(s) \qquad \text{(III-8)}$$

Equation (III-8) gives the relationship between the barred input and barred output. The open-loop barred transfer function is

$$\frac{\overline{C(s)}}{\overline{E(s)}} = R^*(s) \qquad \text{(III-9)}$$

It is evident that the barred functions may be written directly as functions of x, but it is not evident that the same is true for $R^*(s)$.

By definition,

$$R^*(s) = \sum_{n=0}^{\infty} r(nT)\, e^{-nTs} \qquad \text{(III-10)}$$

From (III-4)

$$r(t) = g_R(t)\, u(t) - g_R(t - T)\, u(t - T) \qquad \text{(III-11)}$$

and

$$r(nT) = g_R(nT) - g_R\left[(n - 1)\, T\right] \qquad \text{(III-12)}$$

Substituting (III-12) into (III-10) yields

$$R^*(s) = \sum_{n=0}^{\infty} \left[g_R(nT) - g_R\left[(n - 1)\, T\right]\right] e^{-nTs} \qquad \text{(III-13)}$$

$$R^*(s) = \sum_{n=0}^{\infty} g_R(nT)\, e^{-nTs} - \sum_{n=0}^{\infty} g_R\left[(n-1)T\right]\, e^{-nTs} \qquad \text{(III-14)}$$

Now an index change is made by letting $m = n - 1$ in the second summation. Therefore,

$$R^*(s) = \sum_{n=0}^{\infty} g_R(nT)\, e^{-nTs} - \sum_{m=-1}^{\infty} g_R(mT)\, e^{-(m+1)Ts} \qquad \text{(III-15)}$$

However, $g_R(mT) = 0$ for $m<0$ which alters (III-15) to

$$R^*(s) = \sum_{n=0}^{\infty} g_R(nT)\, e^{-nTs} - \sum_{m=0}^{\infty} g_R(mT)\, e^{-(m+1)Ts} \qquad \text{(III-16)}$$

Since the summations are over the same limits, (III-16) may be written as

$$R^*(s) = \sum_{n=0}^{\infty} g_R(nT) \left[e^{-nTs} - e^{-(n+1)Ts} \right] \qquad \text{(III-17)}$$

From (III-17) and (II-10), it is seen that $R^*(s)$ can be expressed as a function of x_p. In fact, comparing (III-17) with (II-6), it is seen that

$$\mathcal{X}\left[R^*(s)\right] = \mathcal{X}\left[G_R(s)\right]$$

$$= \mathcal{X}\left[\frac{G(s)}{s}\right]$$

$$= G_R(x) \qquad \text{(III-18)}$$

where $\mathcal{X}\left[G(s) \,/\, s\right]$ is a function of x_p only. However, as will be seen, no problems arise from dropping the subscript p in x_p. Then from (III-8),

$$C(x) = E(x)\ G_R(x) \qquad \text{(III-19)}$$

It should be observed from Figure 4 that

$$C(s) = G(s)\ \overline{E(s)} \qquad \text{(III-20)}$$

and therefore in view of (III-19),

$$\overline{C(s)} = \left[\frac{\overline{G(s)}}{s}\right]\ \overline{E(s)} \qquad \text{(III-21)}$$

An example wherein (III-19) is used will clarify the definition of $G_R(x)$.

Examples indicating the use of equation (III-19)

Determination of the output, $C(x)$, for a sampled-hold-data system with $G(s) = \frac{1}{s+1}$ and $T = 1$ second. - It is assumed that a unit step function is applied at the input and the system is that shown in Figure 4. From (III-19)

$$C(x) = E(x)\, G_R(x) \tag{III-22}$$

$$E(x) = \chi\left[\frac{1}{s}\right] = x^0 \tag{III-23}$$

$$G_R(x) = \chi\left[\frac{G(s)}{s}\right] = \frac{1}{s(s+1)}$$

$$= x^0 - \frac{(x^0 - x)}{1 - e^{-T} x} \tag{III-24}$$

or

$$G_R(x) = \frac{0.632\ x}{1 - .368\ x} \tag{III-25}$$

Combining (III-23) and (III-25) according to (III-22) yields

$$C(x) = x^0\left[\frac{0.632\ x}{1 - .368\ x}\right] = \frac{0.632\ x}{1 - .368\ x} \tag{III-26}$$

Using the power series method for taking the inverse of C(x), (III-26) becomes

$$C(x) = 0.632\ x + 0.232\ x^2 + 0.0855\ x^3 + 0.0315\ x^4 + \ldots \quad \text{(III-27)}$$

The inverse of (III-27) is plotted in Figure 5.

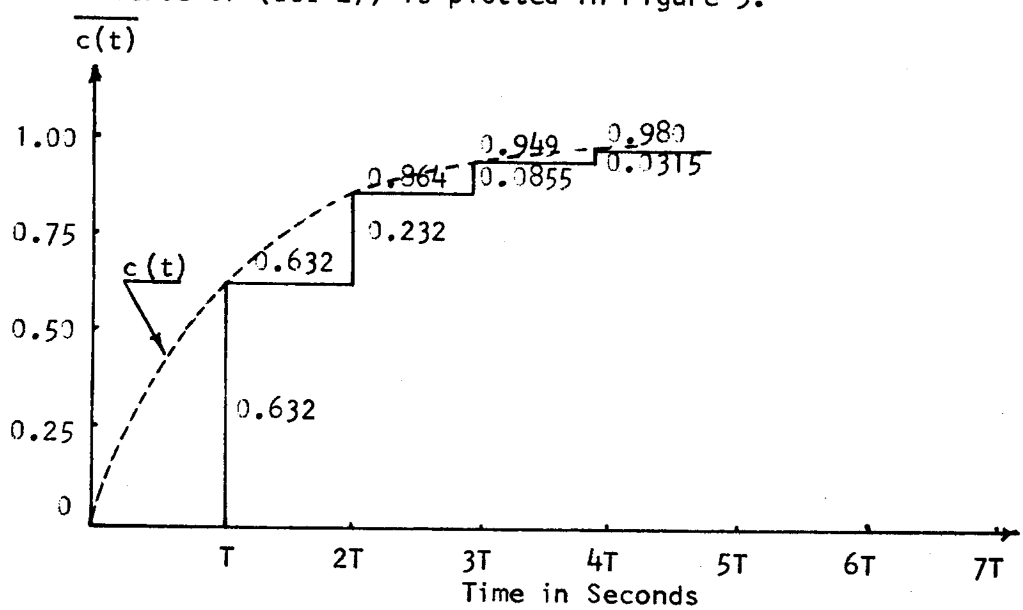


Fig. 5. - Output response of a fictitious sample-hold for the system with G(s) = 1/ (s+1) and a unit step input.

A normal z-transform analysis for this system gives

$$C(z) = 0.632\ z^{-1} + 0.864\ z^{-2} + 0.949\ z^{-3}$$

$$+ 0.980\ z^{-4} + \ldots \quad \text{(III-28)}$$

These values check the x-transform analysis and are also indicated in Figure 5.

Determination of the output, $C(x)$, for a sampled-hold-data system with $G(s) = \frac{1}{s(s+1)}$ and $T = 1$ second. - A unit step function is applied to the input of the system shown in Figure 4. From (III-18)

$$C(x) = E(x)\, G_R(x) \qquad \text{(III-29)}$$

$$E(x) = \mathcal{X}\left[\frac{1}{s}\right] = x^0 \qquad \text{(III-30)}$$

$$G_R(x) = \mathcal{X}\left[\frac{G(s)}{s}\right] = \mathcal{X}\left[\frac{1}{s^2\,(s+1)}\right] \qquad \text{(III-31)}$$

$$= \mathcal{X}\left[t - 1 + e^{-t}\right] \qquad \text{(III-32)}$$

$$= \frac{x}{1 - x} - x^0 + \frac{x^0 - x}{1 - .368\, x} \qquad \text{(III-33)}$$

Combining terms,

$$G_R(x) = \frac{0.264\, x^2 + 0.368\, x}{0.368\, x^2 - 1.368\, x + 1} \qquad \text{(III-34)}$$

Substituting (III-30) and (III-34) into (III-29) yields

$$C(x) = \frac{0.368\, x + 0.264\, x^2}{1 - 1.368\, x + 0.368\, x^2} \qquad \text{(III-35)}$$

Using long division in order to express C(x) in a power series in x gives

$$C(x) = 0.368\ x + 0.768\ x^2 + 0.915\ x^3 + 0.968\ x^4 + \dots \qquad \text{(III-36)}$$

The inverse of (III-36) is plotted in Figure 6.

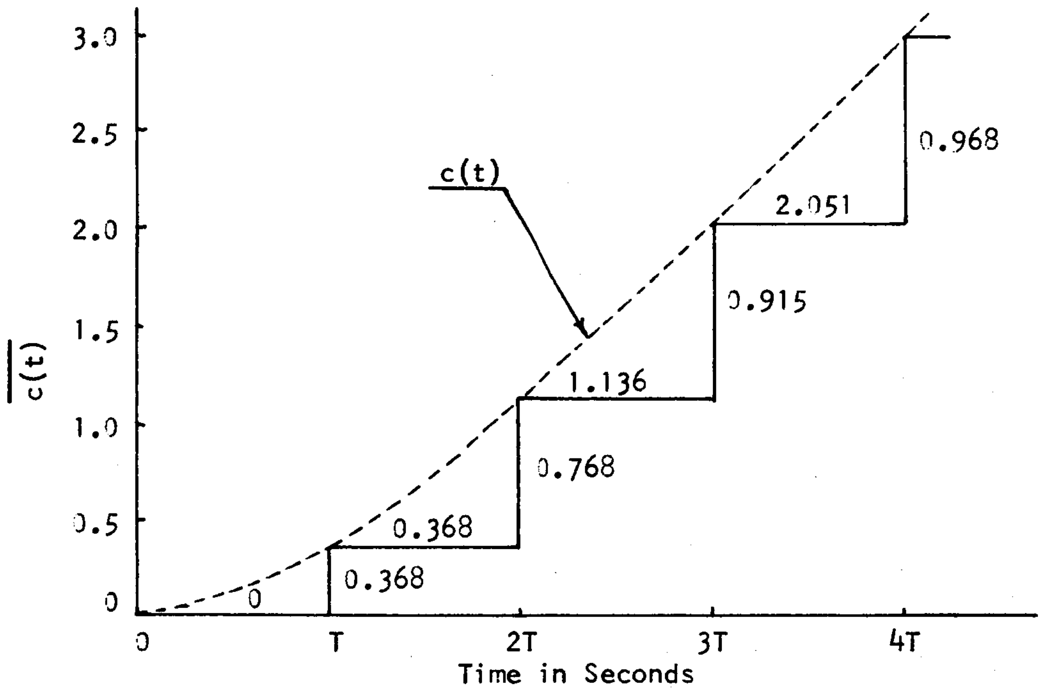


Fig. 6. - Output response of a fictitious sample-hold for the system with G(s) = 1/s(s+1) and a unit step input.

A normal z-transform analysis of this system gives

$$C(z) = 0.368\ z^{-1} + 1.136\ z^{-2} + 2.051\ z^{-3} + \dots \qquad \text{(III-37)}$$

The values for the z-transform analysis are also shown in Figure 6, and they agree with the x-transform evaluation.

Closed-Loop

Derivation of the transfer function

The derivation of the transfer function for a closed-loop sample-hold-data system follows from the block diagram of Figure 7.

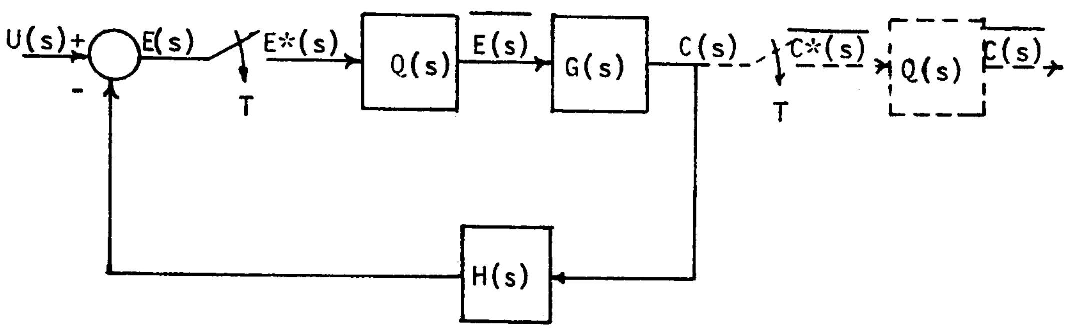


Fig. 7. - Closed-loop sampled-data system.

The closed-loop transfer function is derived as follows:

$$C(s) = G(s)\ \overline{E(s)} \tag{III-38}$$

$$E(s) = U(s) - H(s)\ C(s) \tag{III-39}$$

$$E(s) = U(s) - H(s)\ G(s)\ \overline{E(s)} \tag{III-40}$$

It is seen from (III-20) and (III-21), that when the barred transform of a product of the type $A(s)\ \overline{B(s)}$ is taken, the result is $\overline{[A(s)\ /\ s]}\ \overline{B(s)}$. Thus, barring (III-40) gives

$$\overline{E(s)} = \overline{U(s)} - \left[\frac{\overline{H(s)\ G(s)}}{s}\right] \overline{E(s)} \tag{III-41}$$

Solving for $\overline{E(s)}$ in (III-41), the barred transform of e(t) is given as

$$\overline{E(s)} = \frac{\overline{U(s)}}{1 + \overline{\left[\frac{H(s)\ G(s)}{s}\right]}} \qquad \text{(III-42)}$$

However, from (III-38) and (III-21) the barred transform of the system output c(t) is obtained as

$$\overline{C(s)} = \overline{\left[\frac{G(s)}{s}\right]}\ \overline{E(s)} \qquad \text{(III-43)}$$

Substituting (III-42) into (III-43) yields

$$\overline{C(s)} = \frac{\overline{\left[\frac{G(s)}{s}\right]}\ \overline{U(s)}}{1 + \overline{\left[\frac{H(s)\ G(s)}{s}\right]}} \qquad \text{(III-44)}$$

The x-transform is obtained directly from (III-44). The closed-loop transfer function is given as

$$\frac{C(x)}{U(x)} = \frac{G_R(x)}{1 + (GH)_R\ (x)} \qquad \text{(III-45)}$$

where

$$G_R(x) \triangleq \mathcal{X}\left[\frac{G(s)}{s}\right] \qquad \text{(III-46)}$$

and

$$(GH)_R(x) \triangleq \mathcal{X}\left[\frac{H(s)\ G(s)}{s}\right] \qquad \text{(III-47)}$$

If there is unity feedback, then

$$\frac{C(x)}{U(x)} = \frac{G_R(x)}{1 + G_R(x)} \qquad \text{(III-48)}$$

The work in Chapter IV on stability will be based on (III-45) and systems such as that shown in Figure 7.

<u>Example of x-transform analysis of a closed-loop system</u>

Assume a system such as shown in Figure 7. Let $H(s) = 1$, $G(s) = \dfrac{1}{s(s + 1)}$ and $T = 1$ second. The input is a unit step function. It is desired to determine the sampled-hold output response.

From (III-48)

$$C(x) = \frac{R(x)\ U(x)}{1 + R(x)} \qquad \text{(III-49)}$$

$$G_R(x) = \chi\left[\frac{G(s)}{s}\right] = \chi\left[\frac{1}{s^2\ (s+1)}\right]$$

$$= \frac{0.368\ x + 0.264\ x^2}{1 - 1.368\ x + 0.368\ x^2} \qquad \text{(III-50)}$$

$$U(x) = \chi\left[\frac{1}{s}\right] = x^0 \qquad \text{(III-51)}$$

Substituting (III-50) and (III-51) into (III-49) and simplifying yield

$$C(x) = \frac{0.368\ x + 0.264\ x^2}{1 - 1.00\ x + 0.632\ x^2} \qquad \text{(III-52)}$$

From (III-52) it is found that

$$\begin{aligned} C(x) = {} & 0.368\ x + 0.632\ x^2 + 0.400\ x^3 + 0.000\ x^4 \\ & - 0.253\ x^5 - 0.253\ x^6 - 0.093\ x^7 + 0.067\ x^8 \\ & + 0.128\ x^9 + 0.0859\ x^{10} + 0.0049\ x^{11} \\ & - 0.0493\ x^{12} - 0.0524\ x^{13} + \ldots \end{aligned} \qquad \text{(III-53)}$$

$\overline{c(t)}$ from (III-53) is plotted in Figure 8.

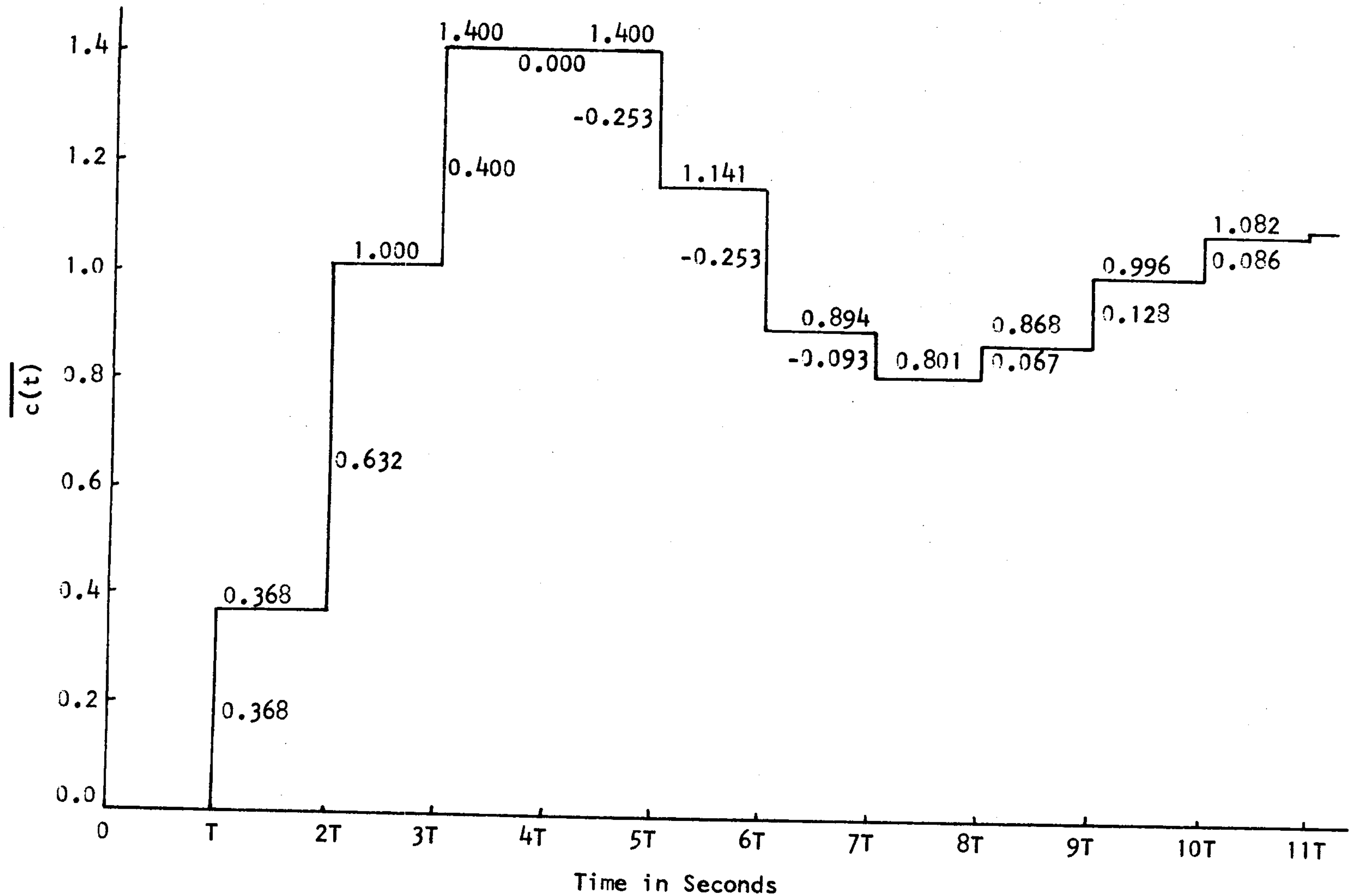


Fig. 8. - Closed-loop response for a sampled-data system with unity feedback and a plant of $G(s) = 1 / s(s + 1)$. $T = 1$ second.

A z-transform analysis of the same system[10] gives

$$C(z) = 0.368\ z^{-1} + 1.00\ z^{-2} + 1.40\ z^{-3} + 1.40\ z^{-4} + 1.15\ z^{-5}$$
$$+ 0.90\ z^{-6} + 0.80\ z^{-7} + 0.86\ z^{-8} + 0.97\ z^{-9} + 1.05\ z^{-10}$$
$$+ 1.06\ z^{-11} + 1.01\ z^{-12} + 0.96\ z^{-13} + \ldots \qquad \text{(III-54)}$$

The values of (III-54) are also shown in Figure 8. The slight differences in (III-53) and (III-54) are the result of the number of significant figures retained in each analysis.

System with Samplers in the Forward Path and the Feedback Path

Consider the case of a sampler-hold in the forward path and a sampler-hold in the feedback path. Such a system is shown in Figure 9.

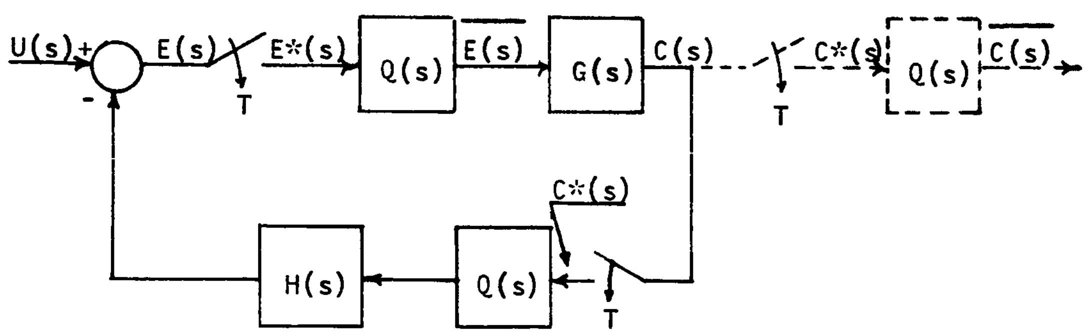


Fig. 9. - Closed-loop sampled-data system with sampled-hold devices in the forward path and the feedback path.

The determination of a transfer function for the system is as follows: From (III-21) the barred output is

$$\overline{C(s)} = \left[\frac{\overline{G(s)}}{s}\right] \overline{E(s)} \tag{III-55}$$

The error signal in Figure 9 is

$$E(s) = U(s) - H(s)\,\overline{C(s)} \tag{III-56}$$

Substituting (III-55) into (III-56) and solving for $\overline{E(s)}$ after barring gives

$$\overline{E(s)} = \frac{\overline{U(s)}}{1 + \left[\frac{\overline{H(s)}}{s}\right] \left[\frac{\overline{G(s)}}{s}\right]} \tag{III-57}$$

Substituting (III-57) into (III-55) and expressing the result as an x-transform give

$$C(x) = \frac{G_R(x)\ U(x)}{1 + H_R(x)\ G_R(x)} \tag{III-58}$$

where

$$H_R(x) = \left[\frac{H(s)}{s}\right] \tag{III-59}$$

For example, consider the system of Figure 9 with $G(s) = 1 / (s + 1)$, $H(s) = 1 / s$, and $U(s) = 1 / s$. The sampling period is assumed to be one second. It is desired to determine the output response, $C(x)$. The solution is as follows:

$$G_R(x) = \chi\left[\frac{1}{s(s+1)}\right]$$

$$= \frac{x - e^{-T}x}{1 - e^{-T}x} \qquad \text{(III-60)}$$

For $T = 1$ second,

$$G_R(x) = \frac{0.632\ x}{1 - 0.368\ x} \qquad \text{(III-61)}$$

Similarly,

$$H_R(x) = \chi\left[\frac{H(s)}{s}\right] = \chi\left[\frac{1}{s^2}\right] = \frac{Tx}{1 - x} \qquad \text{(III-62)}$$

The x-transform of the input is

$$U(x) = \chi\left[\frac{1}{s}\right] = x^0 \qquad \text{(III-63)}$$

Substituting (III-61), (III-62) and (III-63) into (III-58) gives

$$C(x) \;=\; \frac{0.632\ x\ -\ 0.632\ x^2}{1\ -\ 1.368\ x\ +\ x^2} \tag{III-64}$$

Taking the inverse gives

$$C(x) \;=\; 0.632\ x + 0.232\ x^2 - 0.314\ x^3 - 0.662\ x^4 - 0.591\ x^5 - \ldots \tag{III-65}$$

$\overline{c(t)}$ from (III-65) is plotted in Figure 10.

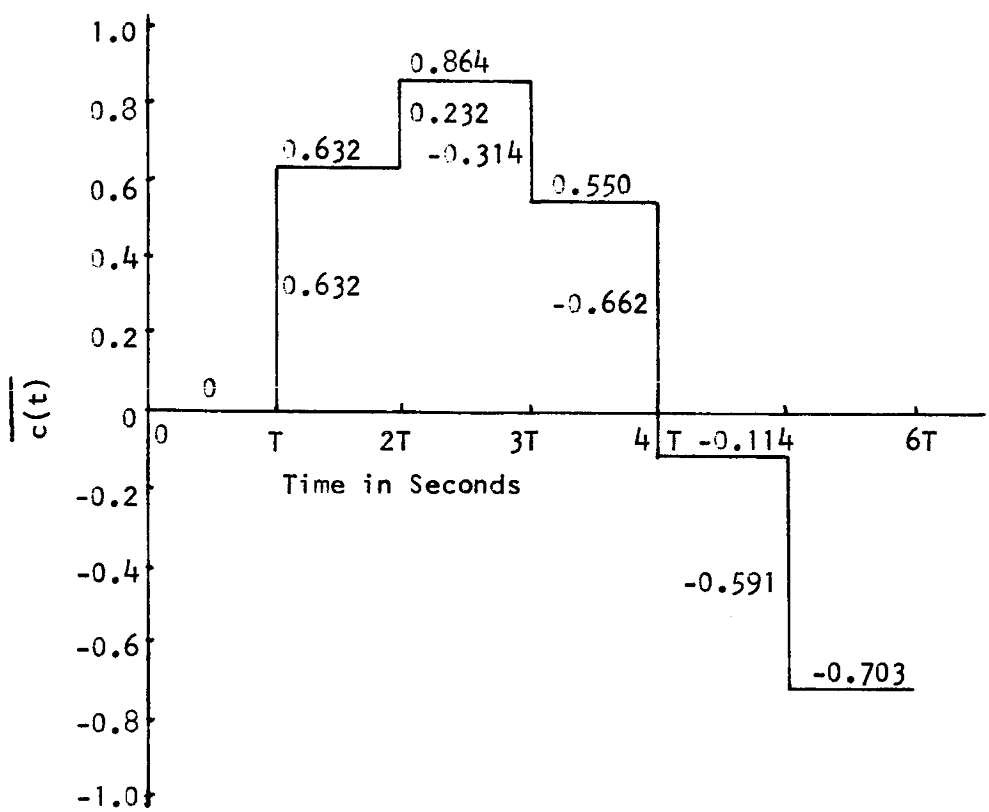


Fig. 10. - System output for the closed-loop system of Fig. 9.

The same system analyzed with the z-transform has an output response of

$$C(z) = 0.632\ z^{-1} + 0.864^{-2} + 0.548^{-3} - 0.114\ z^{-4} - \ldots \quad \text{(III-66)}$$

The values for (III-66) are also indicated in Figure 10. The x-transform analysis and the z-transform analysis give compatible results.

Signal Flow Graphs

Signal flow techniques have been applied to sampled-data systems and these techniques have been described in the literature[11]. The extension of the sampled-data signal flow techniques to sampled-data-hold systems is logical and can be accomplished under the existing rules with the addition of the following rule:

> In taking the barred transform of an equation, all transfer-functions , such as G(s) in the forward path of H(s) in the feedback path, are divided by s and then barred.

Under this rule, the x-transform is obtained directly from the barred function.

It should be remembered that in the barred notation or the x-domain, the sampler and zero-order hold are treated as an entity. The steps involved in obtaining the signal flow graph are:

(1) With the system block diagram as the starting point, the "original signal flow graph" of the system is constructed.

(2) From the "original signal flow graph" the equations describing the "sampled signal flow graph" are obtained.

(3) The "sampled signal flow graph" is constructed from the describing equations in (2).

As an example, consider the system shown in Figure 11. The

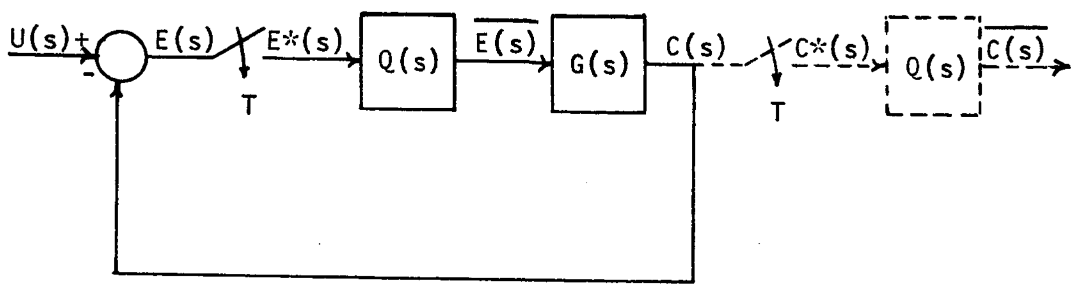


Fig. 11. - Closed-loop sampled-data system.

"original signal flow graph" is constructed in the usual manner as shown in the lower section of Figure 12. From this "original signal

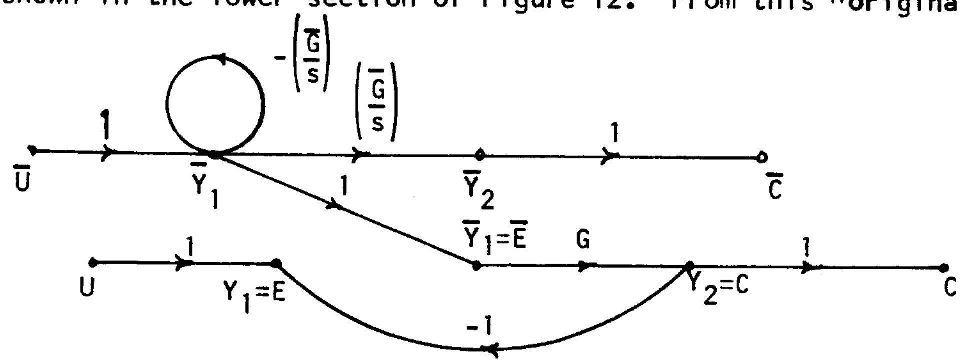


Fig. 12. - Composite signal flow graph of the sampled-data system shown in Fig. 11.

flow graph," the following equations are obtained:

$$Y_1 = U - \overline{Y_1 \; G} \tag{III-67}$$

$$Y_2 = \overline{Y_1} \; G \tag{III-68}$$

where the "function of s" notation in each term has been omitted for simplicity. Taking the barred transform of (III-67) and (III-68) yields

$$\overline{Y_1} = \overline{U} - \overline{Y_1}\left[\frac{\overline{G}}{s}\right] \tag{III-69}$$

$$\overline{Y_2} = \overline{Y_1}\left[\frac{\overline{G}}{s}\right] \tag{III-70}$$

The "sampled-data signal flow graph" is drawn from (III-69) and (III-70) and appears as the upper graph in Figure 12. In this example the common node on the two graphs is $\overline{Y_1}$ and the graphs are connected at this common node. The composite signal flow graph of the sampled-data system of Figure 11 is now complete.

From the composite signal flow graph the barred output is obtained by the use of Mason's gain formula as

$$\overline{C(s)} = \frac{\left[\dfrac{\overline{G(s)}}{s}\right]\overline{U(s)}}{1 + \left[\dfrac{\overline{G(s)}}{s}\right]} \tag{III-71}$$

or

$$C(x) = \frac{G_R(x)\ U(x)}{1 + G_R(x)} \tag{III-72}$$

The continuous output is

$$C(s) = \frac{G(s)\,\overline{U(s)}}{1 + \left[\dfrac{\overline{G(s)}}{s}\right]} \tag{III-73}$$

As a second example, consider the system of Figure 9. The composite signal flow graph is shown in Figure 13. From Figure 13

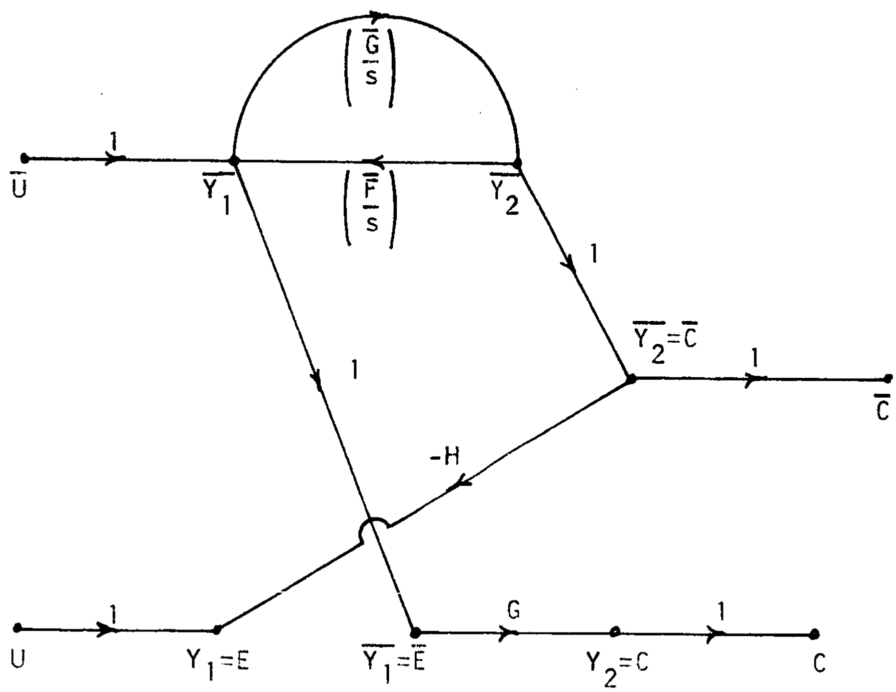


Fig. 13 - Composite signal flow graph for the system of Fig. 9.

the barred output is obtained as

$$\overline{C(s)} = \frac{\overline{U(s)}\left[\dfrac{\overline{G(s)}}{s}\right]}{1 + \left[\dfrac{\overline{G(s)}}{s}\right]\left[\dfrac{\overline{H(s)}}{s}\right]} \tag{III-74}$$

or

$$C(x) = \frac{U(x)\ G_R(x)}{1 + G_R(x)\ H_R(x)} \tag{III-75}$$

Equation (III-75) checks (III-58) which was derived directly from the block diagram. The continuous output is

$$C(s) = \frac{\overline{U(s)}\ G(s)}{1 + \left[\dfrac{\overline{G(s)}}{s}\right] \left[\dfrac{\overline{F(s)}}{s}\right]} \tag{III-76}$$

IV. STABILITY OF SAMPLED-HOLD-DATA SYSTEMS

In the terminology of a linear continuous-data system the definition of stability is given as:

> A system is stable if the output response to any bounded input disturbance is finite.[12]

It would seem that this same definition could apply to sampled-data-hold systems since their outputs are piece-wise continuous. However, the analysis of such systems is being accomplished through the x-transform which, although it gives a continuous output, is a "jump" analysis. The output $\overline{c(t)}$ is equal to $c(t)$ only at the sampling instants, $c(nT)$. Therefore, any stability tests on sampled-data-hold systems will be conducted with respect to the sampled output rather than the actual output.

There must be an element of caution exercised in the application of x-transform stability analysis. If the system response contains hidden oscillations[13], then the x-transform method of stability analysis will lead to erroneous results.

Since all of the systems under analysis in this dissertation contain a hold device, it is expected that the stability problem will be more acute than in systems without such hold devices. The hold circuit is equivalent to adding phase lag in the system and phase lag is likely to have an adverse effect on the stability of feedback control systems.

Consider the continuous and sampled-data systems shown in Figure 14. For the continuous system to be stable, all of the poles

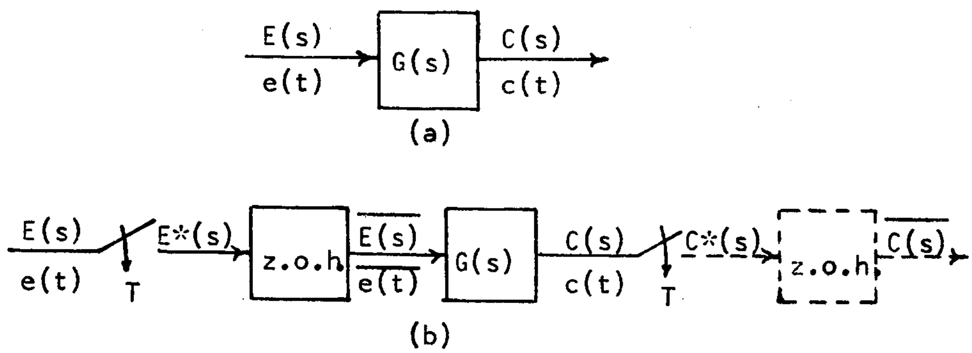


Fig. 14. - Continuous- and sampled-data systems.

of G(s) must lie in the left half of the s-plane; if any pole of G(s) lies in the right half of the s-plane, the system is unstable. This can readily be seen by taking the inverse Laplace transform of G(s); a pole in the right half of the s-plane means that it has a positive real part which indicates an ever-increasing exponential in the time domain. Suppose that the sampled-data system of Figure 14b has no hidden oscillations and that the x-transform method of analysis is applicable. It can be said that such a system is stable if the output response $\overline{c(t)}$ is bounded for a bounded input. This statement must be investigated concerning the placement of poles in the x-plane.

If in the definition of the x-transform, s is replaced by $\sigma + j\omega$, the transformation of the stability boundary in the s-plane onto the x-plane may be observed. Hence,

$$x = \frac{e^{-(\sigma + j\omega)T}}{\sigma + j\omega} \qquad \text{(IV-1)}$$

Letting $\sigma = 0$ and ω take on values from zero to infinity, it is found that the $j\omega$ axis in the s-plane transforms onto a spiral in the x-plane as shown in Figure 15. Furthermore, the shape of this spiral depends on the sampling period T.

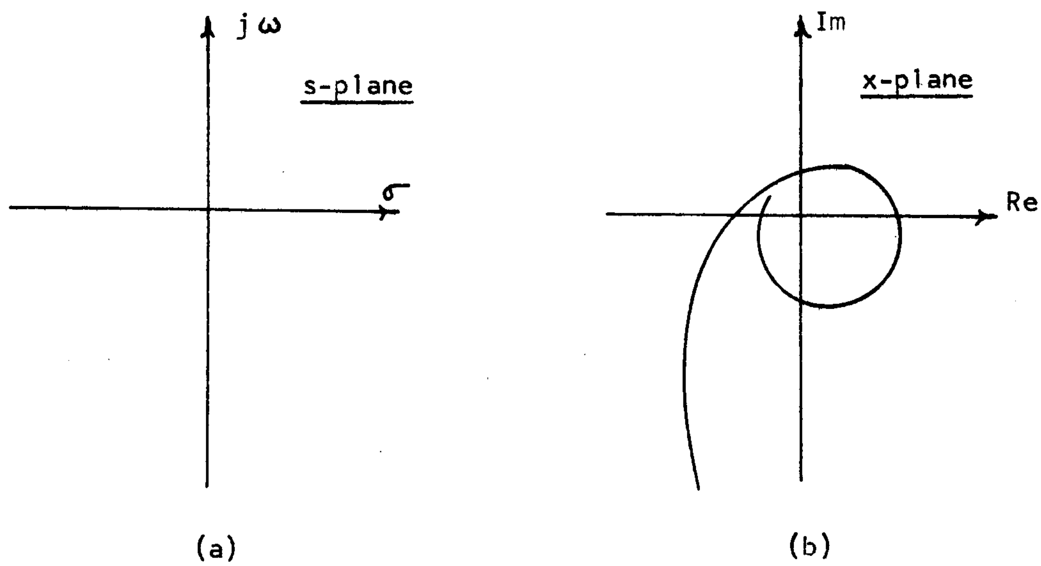


Fig. 15. - Transforming the $j\omega$ axis of the s-plane onto the x-plane.

This means that the actual use of the x-plane for stability studies is undesirable. However, for closed-loop systems, the stability depends on the location of the roots of the characteristic equation

$$1 + \overline{(GH)_R\ (s)} = 0 \tag{IV-2}$$

It must be remembered that R(s) is not G(s) but

$$R(s) = Q(s)\ G(s) = \left[\frac{1 - e^{-Ts}}{s}\right] G(s) \tag{IV-3}$$

or

$$R^*(s) = \sum_{n=0}^{\infty} g_R(nT) \left[e^{-nTs} - e^{-(n+1)Ts}\right] \qquad \text{(IV-4)}$$

where

$$G_R(s) \triangleq \frac{G(s)}{s} \qquad \text{(IV-5)}$$

By definition

$$\overline{(GH)_R(s)} \triangleq \left[\frac{\overline{G(s)\ H(s)}}{s}\right] \qquad \text{(IV-6)}$$

Thus, in light of (IV-2) and (IV-4), coupled with the indication in (III-18), a transfer function in the x-domain is a function only of the pseudo x-transform, x_p, which is defined as

$$x_p \triangleq e^{-Ts} \qquad \text{(IV-7)}$$

and the stability analysis can be performed in the x_p-plane. Using the x_p notation, (IV-2) can be written as

$$1 + (GH)_R\ (x_p) = 0 \qquad \text{(IV-8)}$$

The stability boundary in the x_p-plane must be determined. It is known that the $j\omega$ axis in the s-plane maps onto a unit circle centered at the origin in the z-plane. From (IV-7) it is observed that

$$x_p = \frac{1}{z} \qquad \text{(IV-9)}$$

The transformation (IV-9) sets up a one to one correspondence between points in the z-plane and points in the x_p-plane, except for the points $z = 0$ and $x_p = 0$, which have no images.

In polar coordinates (IV-9) becomes

$$\rho e^{j\phi} = \frac{1}{r} e^{-j\theta} \qquad \text{(IV-10)}$$

Equation (IV-10) can be described by the consecutive transformations

$$z' = \frac{1}{r} e^{j\theta}, \quad x_p = z' \text{ conjugate} \qquad \text{(IV-11)}$$

The first transformation in (IV-11) is an inversion with respect to the unit circle $r = 1$ (See Figure 16). The point z' lies on a

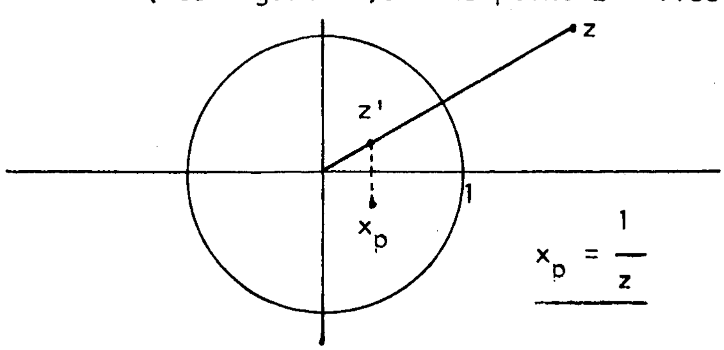


Fig. 16. - Transforming the z-plane onto the x_p-plane.

radius drawn through the point z, and its distance from the center is such that

$$|z'| \; |z| = 1 \tag{IV-12}$$

The second transformation in (IV-11) reflects z' across the real axis.

This says that points outside the unit circle are mapped into points inside the unit circle and conversely. Points on the unit circle are simply reflected across the real axis.

From the preceding it is seen that the stability boundary in the x_p-plane is the unit circle centered at the origin. Therefore the stability analysis can be accomplished completely in the x_p-plane provided that the results are interpreted as follows (see Figure 17):

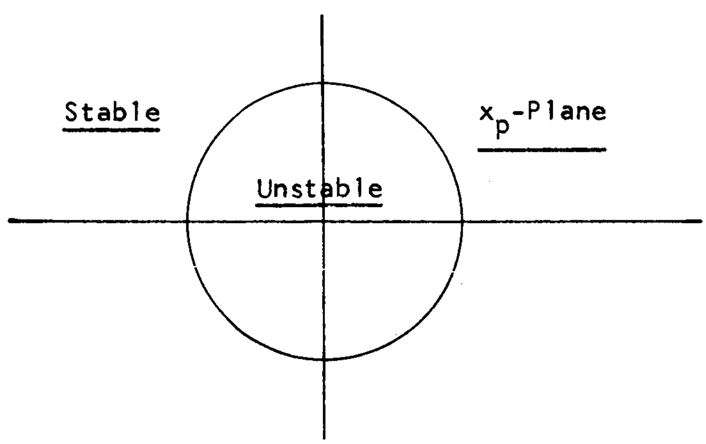


Fig. 17. - Pseudo x-plane showing stable and unstable regions.

(1) Points outside the unit circle are in the stable region.

(2) Points inside the unit circle are in the unstable region.

(3) Points on the unit circle indicate sustained oscillations.

Stability of Sampled-Data Systems Through the X_p-Transform Analysis

It has been shown that x_p maps the left half of the s-plane onto the exterior of the unit circle $|x_p| = 1$ and it maps the right half of the s-plane onto the interior of the unit circle $|x_p| = 1$. The important definitions in this mapping are repeated as

$$x_p \overset{\Delta}{=} \text{pseudo } x \overset{\Delta}{=} e^{-sT} \tag{IV-13}$$

$$z \overset{\Delta}{=} e^{sT} \tag{IV-14}$$

$$x_p = \frac{1}{z} \tag{IV-15}$$

Then for $s = \sigma + j\omega$,

$$x_p = e^{-\sigma T} e^{-j\omega T} \tag{IV-16}$$

If $\sigma = 0$, the $j\omega$ axis of the s-plane maps onto the x_p-plane as a unit circle in the manner shown in Figure 18.

It should be observed from Figure 18 that traversing up the $j\omega$ axis (ω increasing) is equivalent to going clockwise around the unit circle in the x_p plane.

It can be stated now that the necessary and sufficient condition for a sampled-data feedback system (where it is understood that all

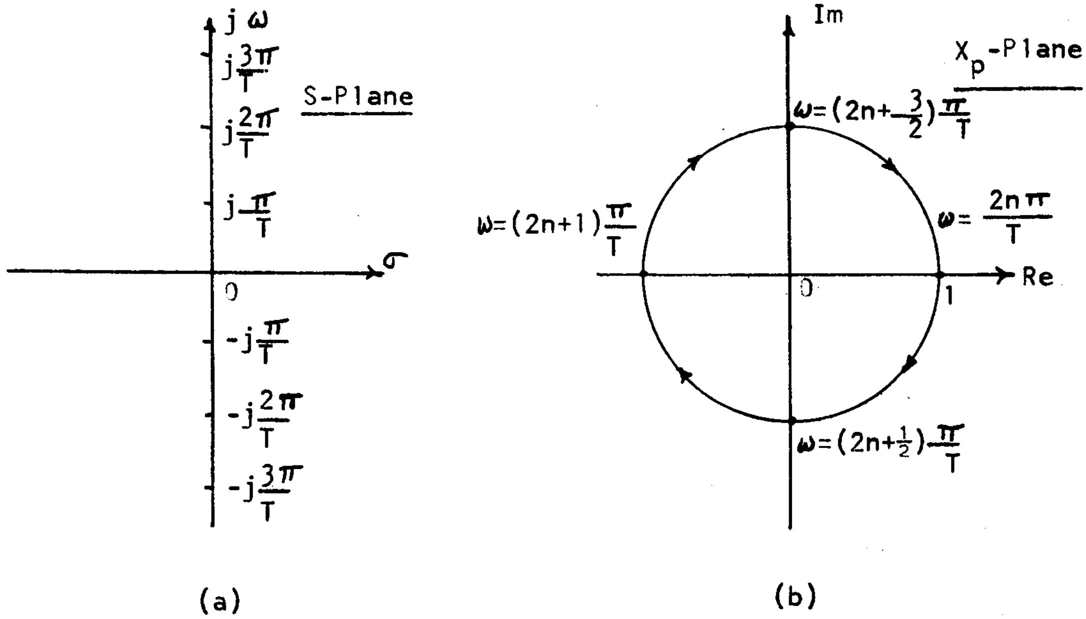


Fig. 18. - Mapping the $j\omega$ axis of the s-plane onto the x_p-plane.

samplers are followed by zero-order holds) to be stable is that all the poles of the over-all transfer function, $C(x) / U(x)$, which is a function of only x_p, lie outside the unit circle in the x_p-plane.

As an alternate statement: The necessary and sufficient condition for a sampled-data feedback system (same configuration as above) to be stable is that all the roots of its characteristic equation in x_p must have an absolute value greater than one.

The stability tests that will be investigated under the x_p-transform method are:

(1) The Routh-Hurwitz Criterion

(2) The Nyquist Criterion

(3) The Bode plot

(4) The Gain-Phase plot

(5) The Root-Locus plot

The Routh-Hurwitz Criterion

In attempting to apply the conventional Routh-Hurwitz criterion to sampled-data systems, it is seen that difficulties arise immediately. If the barred notation is used, the complex variable s appears and the equations are transcendental in s. Routh-Hurwitz applies only to algebraic equations[14].

The following definition from Sokolnikoff and Redheffer[15] might aid in understanding the problem.

> A polynomial equation $y^n + a_1 y^{n-1} + \ldots + a_n = 0$ is called an algebraic equation. An equation $F(y) = 0$ which is not reducible to an algebraic equation is called transcendental. Thus, $\tan y - y = 0$ is a transcendental equation, and so is $e^y + 2 \cos y = 0$.

Difficulty also exists in the x_p-plane because the boundary of stability is the unit circle $|x_p| = 1$ and not the imaginary axis.

However, this problem can be circumvented by mapping the interior of the unit circle in the x_p-plane onto the right-half plane of some other complex variable plane by a bilinear transformation such as:

$$x_p = \frac{1 - \alpha}{1 + \alpha} \quad \text{or} \quad x_p = \frac{\beta - 1}{\beta + 1} \tag{IV-17}$$

Either transformation maps the interior of the unit circle in the x_p-plane onto the right half of the α-plane or β-plane, as the case may be. For a given value of x_p,

$$\alpha = \frac{1}{\beta} \tag{IV-18}$$

Once the transformation has been accomplished, the Routh-Hurwitz criterion may be applied directly to the new equation in the variable α or β.

As an example of the application of the Routh-Hurwitz criterion to a sampled-data system, consider the system shown in Figure 19.

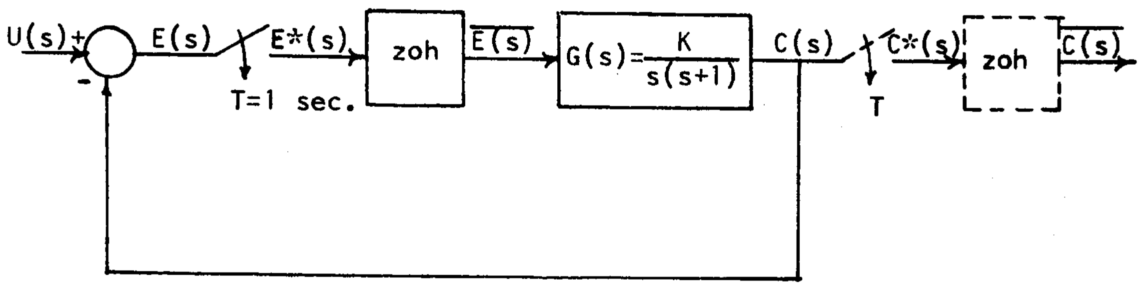


Fig. 19. - A sampled-data system with unity feedback.

It is desired to determine the limits on K for stability. The characteristic equation is

$$1 + G_R(x_p) = 0 \tag{IV-19}$$

$$G_R(x_p) = \mathcal{X}\left[\frac{G(s)}{s}\right]$$

$$= K\left[\frac{0.264\, x_p^2 + 0.368\, x_p}{0.368\, x_p^2 - 1.368\, x_p + 1}\right] \tag{IV-20}$$

Substituting (IV-20) into (IV-19) and simplifying yield

$$(0.368 + 0.264\ K)\ x_p^2 + (0.368\ K - 1.368)\ x_p + 1 = 0 \qquad \text{(IV-21)}$$

Substituting the β transformation of (IV-17) into (IV-21) and simplifying give

$$0.632\ K\ \beta^2 + (1.264 - 0.528\ K)\beta + (2.736 - 0.104\ K) = 0 \qquad \text{(IV-22)}$$

The Routh's array, which is determined from (IV-22), is

$$\begin{array}{lll} \beta^2 & 0.632\ K & (2.736 - 0.194\ K) \\ \beta^1 & (1.264 - 0.528\ K) & \\ \beta^0 & (2.736 - 0.104\ K) & \end{array}$$

The Routh-Hurwitz criterion states that a system is stable if the elements in the first column of the Routh's array are all positive (or all negative). Therefore, for the system of Figure 19 to be stable, the following conditions on K are required:

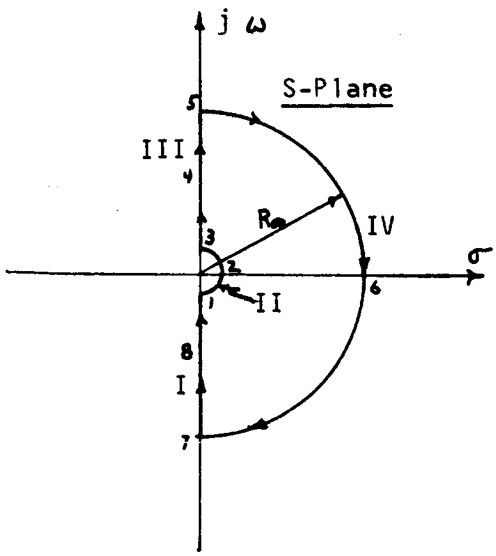
$$\begin{array}{l} K > 0 \\ K < 2.4 \end{array} \qquad \text{(IV-23)}$$

It may be said that the application of the Routh-Hurwitz criterion in stability studies of sampled-data systems is straightforward. The feasibility of the method, which depends on the use of a bilinear transformation, is determined by the order of the system under study. Higher order systems may require more labor than the result warrants. It is well to remember that the Routh-Hurwitz criterion tells nothing about the degree of stability.

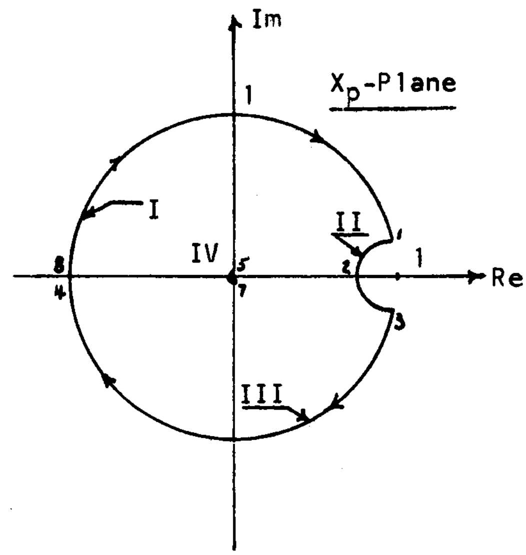
The Nyquist Criterion

The Nyquist path

The Nyquist path in the s-plane for continuous-data systems is shown in Figure 20a. The Nyquist path in the x_p-plane is shown in Figure 20b.



(a)



(b)

Fig. 20. - The Nyquist path in the (a) s-plane (b) x_p-plane.

The x_p-transform method

The Nyquist plot of $G_R(x_p)$ is a plot of $G_R(x_p)$ as x_p takes on values along the Nyquist path in the x_p-plane, which is the unit circle with center at the origin.

The net number of rotations, N, of $1 + G_R(x_p)$ about the origin of the $1 + G_R(x_p)$-plane is equal to its total number of poles, P, minus its total number of zeros, Z, inside the unit circle in the x_p-plane. Thus,

$$N = P - Z$$

where counterclockwise rotation is defined as being positive and clockwise rotation as being negative.

As is normally done in a Nyquist analysis, the critical point is made at (-1, 0) instead of the origin. This is accomplished by shifting the imaginary axis one unit to the right and observing $G_R(x_p)$ rather than $1 + G_R(x_p)$.

For a stable closed-loop sampled-data system, the Nyquist criterion states that the Nyquist plot for $G_R(x_p)$ will encircle the (-1, 0) point of the $G_R(x_p)$-plane in a counterclockwise direction such that the net number of encirclements will be equal to the number of poles P that lie inside the Nyquist path in the x_p-plane. Thus,

$$N = P$$

If the system is open-loop stable (P = 0), then the criterion simplifies to

$$N = 0$$

and $G_R(x_p)$ should not encircle the (-1, 0) point at all.

It is concluded from the preceding statements that the application of the Nyquist criterion to a sampled-data system is an investigation of the behavior of the Nyquist plot of $G_R(x_p)$ with respect to the (-1, 0) point. If the plot can be constructed, then the application can be made.

As an example, consider the sampled-hold-data system with a loop transfer function of

$$G(s) = \frac{K}{s(s+1)} \tag{IV-24}$$

For T = 1 second

$$G_R(x) = \chi\left[\frac{G(s)}{s}\right] = \chi\left[\frac{K}{s^2\,(s+1)}\right] \tag{IV-25}$$

or

$$G_R(x) = K\left[\frac{0.264\,x^2 - 0.368\,x}{(1-x)(1-0.368\,x)}\right] \tag{IV-26}$$

Since $G_R(x)$ is actually a function of x_p only, one obtains

$$\frac{G_R(x_p)}{K} = \frac{0.264\,x_p^2 + 0.368\,x_p}{(1-x_p)(1-0.368\,x_p)} \tag{IV-27}$$

From Figure 20b the relations between points on the unit circle in the x_p-plane and points on the $j\omega$ axis in the s-plane are obtained as

$$
\begin{aligned}
x_p &= 1 \angle -90^\circ & \omega &= n\omega_s + \frac{\omega_s}{4} \\
x_p &= 1 \angle -180^\circ & \omega &= n\omega_s + \frac{2\omega_s}{4} \\
x_p &= 1 \angle -270^\circ & \omega &= n\omega_s + \frac{3\omega_s}{4} \\
x_p &= 1 \angle -360^\circ & \omega &= n\omega_s + \frac{4\omega_s}{4}
\end{aligned}
\qquad \text{(IV-28)}
$$

Evaluating (IV-27) by (IV-28) gives

$$
\begin{aligned}
K^{-1}\, G_R(-j) &= 0.3 \angle -191^\circ \\
K^{-1}\, G_R(-1) &= -0.038 \\
K^{-1}\, G_R(j) &= 0.3 \angle -169^\circ \\
K^{-1}\, G_R(1) &= \infty
\end{aligned}
\qquad \text{(IV-29)}
$$

The points (IV-29) along with some supplementary points are plotted in Figure 21.

In order to complete the Nyquist plot of $G_R(x_p)$, the section of the Nyquist path in the x_p-plane corresponding to Section II

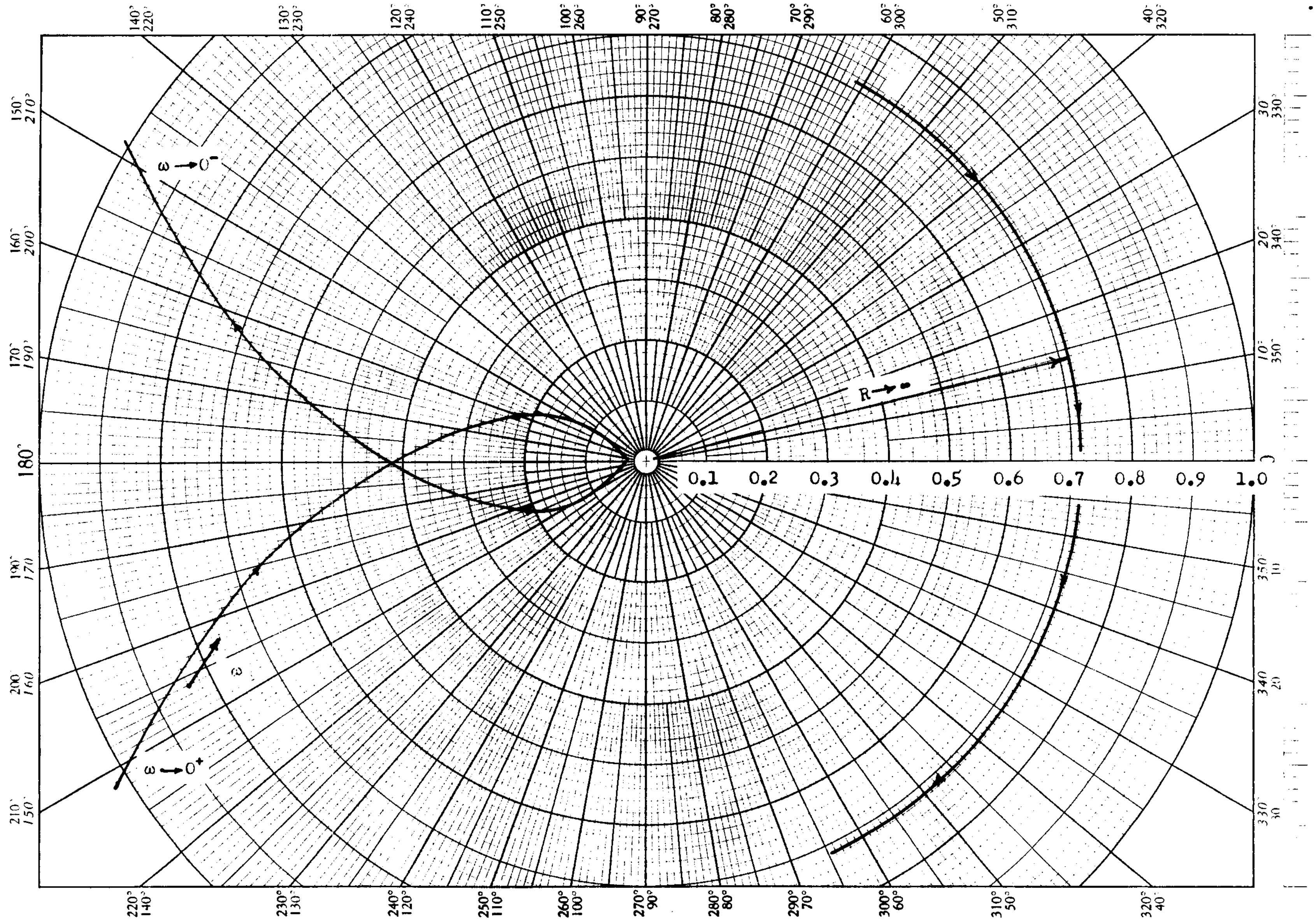


FIG. 21. - Nyquist plot for $K^{-1}G_R(x_p) = \dfrac{0.264x_p^2 + 0.368x_p}{(1 - x_p)(1 - 0.368x_p)}$. $G(s) = \dfrac{K}{s(s + 1)}$.

(points 1, 2, 3 in Figure 20b) should be considered. Figure 22 gives

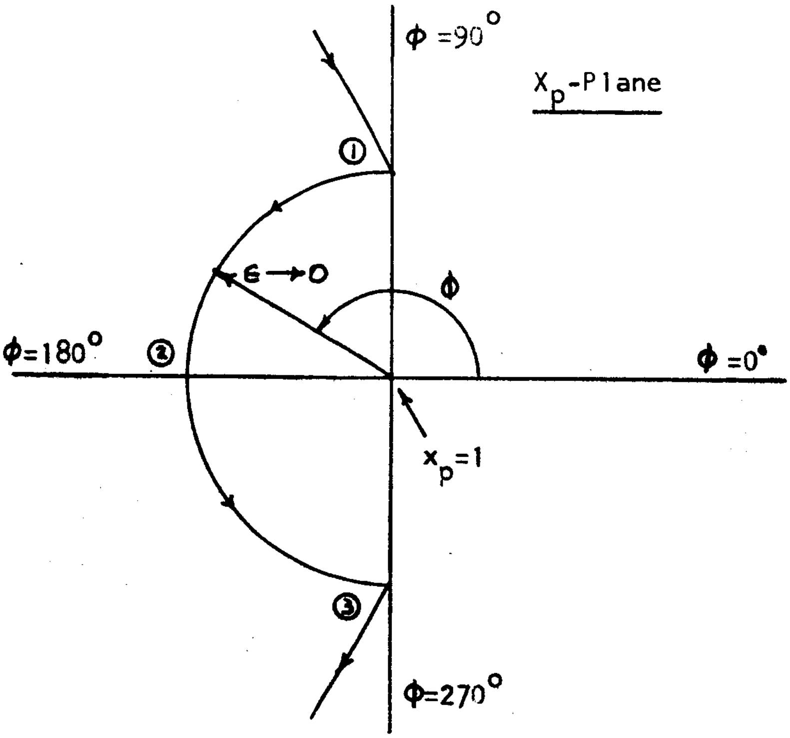


Fig. 22. - Section II of the Nyquist path in the x_p-plane.

an enlarged view of the indention in the Nyquist path. The points in this section may be represented by

$$x_p = 1 + \epsilon e^{j\phi} \tag{IV-30}$$

where ϵ tends to zero and $90^\circ \leq \phi \leq 270^\circ$. Substituting (IV-30) into (IV-27) gives

$$K^{-1} G_R(x_p) = \frac{\left[0.264 \ (1 + 2\epsilon e^{j\phi} + \epsilon^2 e^{j2\phi}) + 0.368 + 0.368 \ \epsilon \ e^{j\phi} \right]}{- \epsilon \ e^{j\phi} \ (0.632 - \epsilon \ e^{j\phi})} \tag{IV-31}$$

Since $\epsilon \longrightarrow 0$, then (IV-31) may be written as

$$K^{-1} G_R(x_p) = - \frac{1}{\epsilon} e^{-j\phi} \tag{IV-32}$$

Therefore, the magnitude of $K^{-1} G_R(x_p)$ on Section II of the Nyquist path approaches infinity and as ϕ varies from 90° through $+180^\circ$ to 270° in a counterclockwise direction, $K^{-1} G_R(x_p)$ varies according to (IV-32) through a rotation of 180° in a clockwise direction about the origin of the $G_R(x_p)$-plane. This section of the Nyquist plot is also shown in Figure 21.

It should be noted from Figure 21 that the critical value of K is

$$K_c = \frac{1}{0.41} = 2.44 \tag{IV-33}$$

This value checks with the value obtained through the Routh-Hurwitz analysis of this same system. The gain-and phase-margins for this system as determined from Figure 21 are 7.5db and 31° respectively.

<u>The bilinear transformation method</u>

The same plot obtained in Figure 21 may be obtained under the bilinear transformations

$$x_p = \frac{\beta - 1}{\beta + 1} \quad \text{or} \quad x_p = \frac{1 - \alpha}{1 + \alpha} \tag{IV-34}$$

Substituting (IV-34) into (IV-27) and simplifying give

$$K^{-1} G_R \left[x_p = \frac{\beta - 1}{\beta + 1} \right] = \frac{0.5(\beta - 1.0)(\beta + 0.165)}{(\beta + 2.16)} \qquad \text{(IV-35)}$$

Let $\beta = j\omega_\beta$, where ω_β is the imaginary part of β. Then,

$$K^{-1} G_R(j\,\omega_\beta) = \frac{-0.0382\,(1 - j\,\omega_\beta)(1 + j\,6.06\,\omega_\beta)}{1 + j\,0.463\,\omega_\beta} \qquad \text{(IV-36)}$$

Plotting (IV-36) gives the results shown in Figure 23, which is the same plot as shown in Figure 21.

Relationship between frequencies

Using the bilinear transformation

$$x_p = \frac{\beta - 1}{\beta + 1} \qquad \text{(IV-37)}$$

one finds that

$$\beta = \frac{1 + x_p}{1 - x_p} \qquad \text{(IV-38)}$$

However, for real frequencies

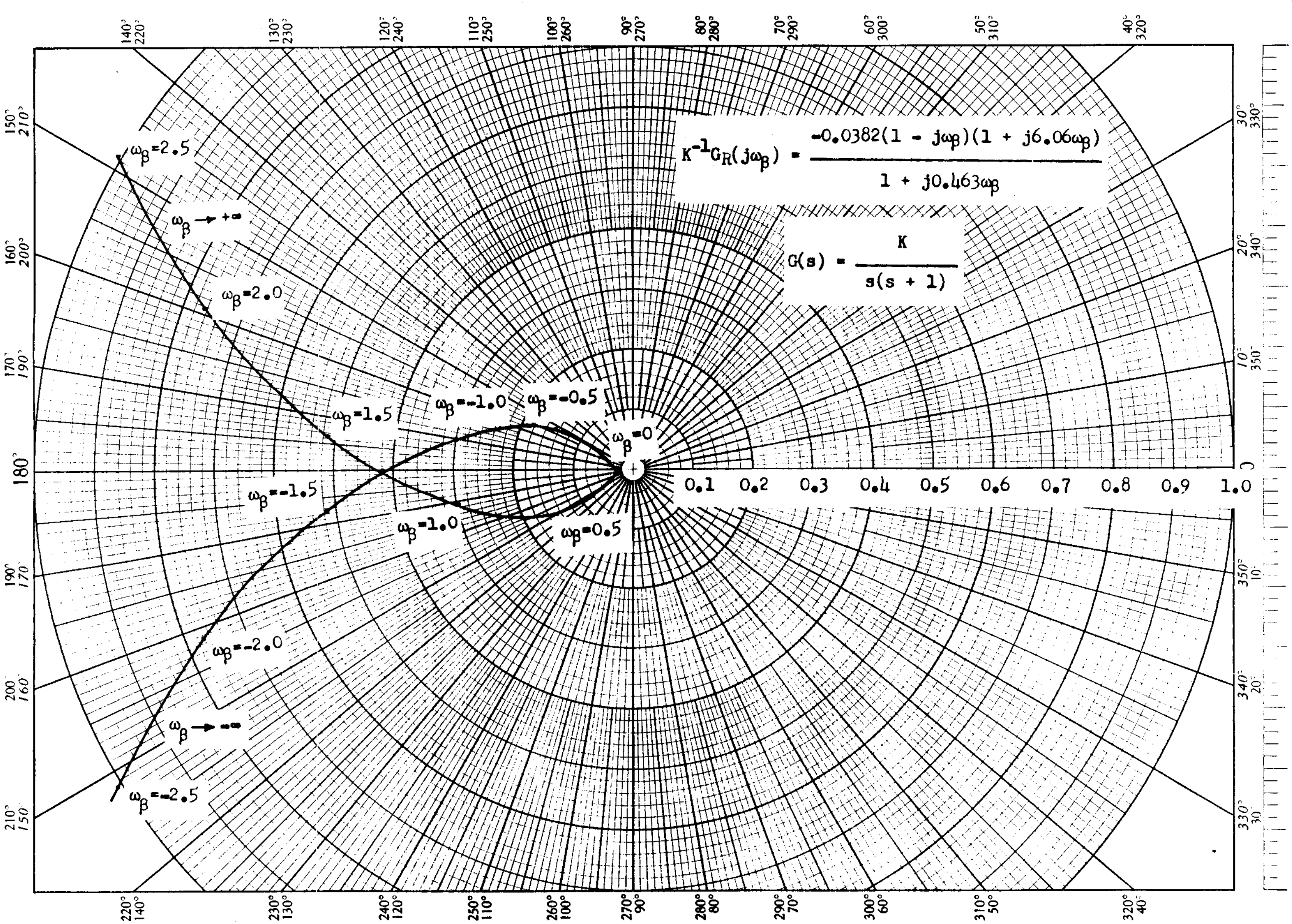


FIG. 23. - Nyquist plot using β transformation.

$$x_p = e^{-j\omega T} \tag{IV-39}$$

and

$$\beta = j\omega_\beta \tag{IV-40}$$

Substituting (IV-39) and (IV-40) into (IV-38) gives

$$j\omega_\beta = \frac{1 + e^{-j\omega T}}{1 - e^{-j\omega T}} \tag{IV-41}$$

Simplifying (IV-41) gives

$$\omega_\beta = -\cot\left[\frac{\omega T}{2}\right] \tag{IV-42}$$

In a similar manner it may be shown that

$$\omega_\alpha = \tan\left[\frac{\omega T}{2}\right] \tag{IV-43}$$

Equations (IV-42) and (IV-43) give the relationships between ω, ω_β, and ω_α. Implicitly included is $\omega_s = 2\pi/T$.

The Bode Diagram

The Bode diagram is constructed in the same manner as in continuous data systems once the bilinear transformation has been made.

As an example, consider the system described by (IV-24) which has an open-loop transfer function of

$$G(s) = \frac{K}{s(s + 1)} \tag{IV-44}$$

Under the bilinear transformation $x_p = (\beta - 1) / (\beta + 1)$, the equation from which the Bode plot is constructed is

$$K^{-1} G_R(j\,\omega_\beta) = \frac{-0.0382\ (1 - j\,\omega_\beta)(1 + j6.06\,\omega_\beta)}{1 + j\,0.463\,\omega_\beta} \tag{IV-45}$$

where T has been assumed one and $\beta = j\omega_\beta$. From (IV-45) the corner frequencies are at ω_β = 2.16, 1.0, and 0.165. The dc gain is -28.36db. The Bode plot is shown in Figure 24. The marked indications of gain margin and phase margin are equal to those obtained from the Nyquist diagram of the same system (See page 73).

The Gain-Phase Plot

The gain-phase diagram may be constructed directly from the loop-gain expression or from the Bode plot. It is a diagram for the open-loop transfer function, $G_R(x_p)$, magnitude as a function of phase

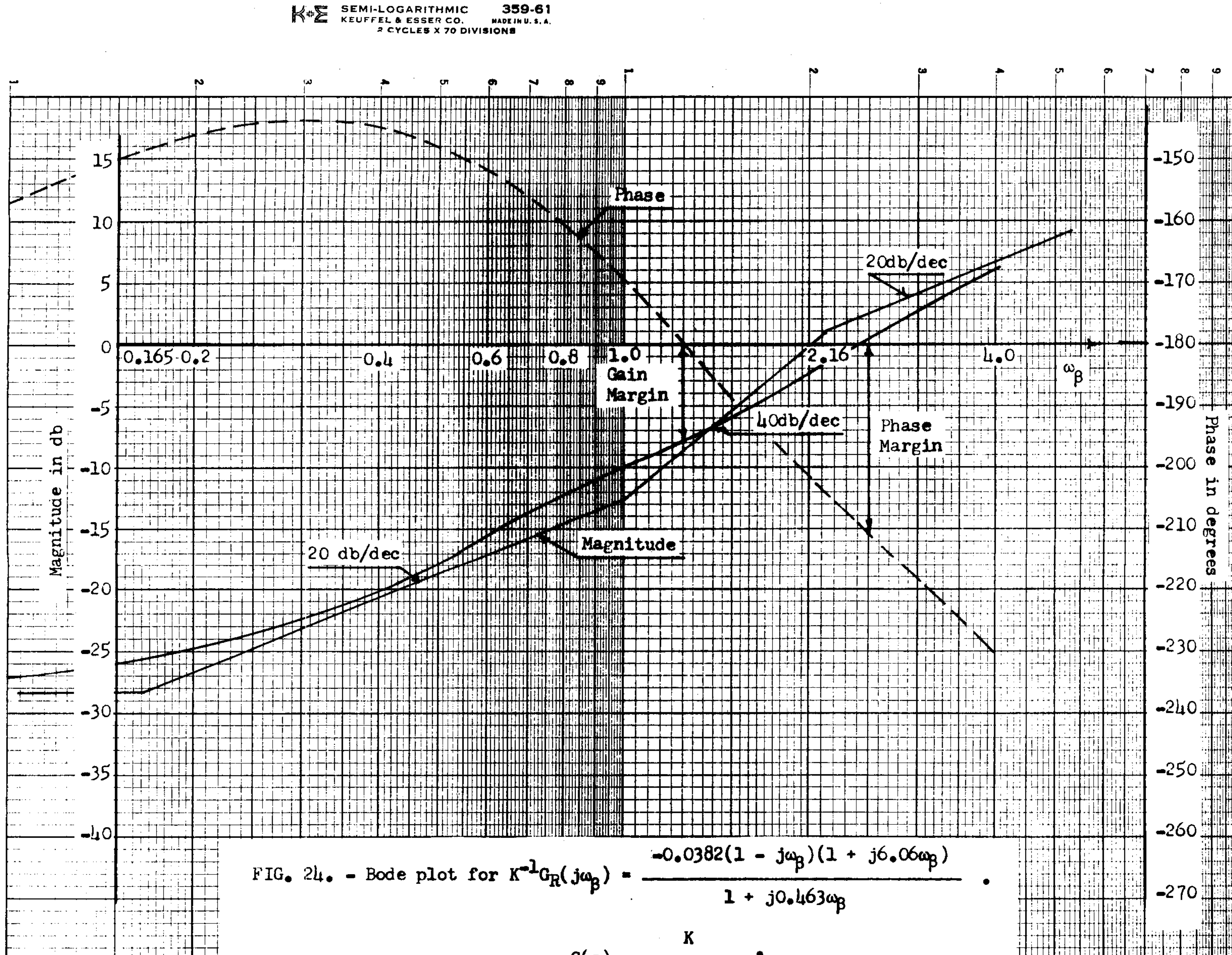


FIG. 24. - Bode plot for $K^{-1}G_R(j\omega_\beta) = \dfrac{-0.0382(1 - j\omega_\beta)(1 + j6.06\omega_\beta)}{1 + j0.463\omega_\beta}$.

$$G(s) = \frac{K}{s(s+1)}.$$

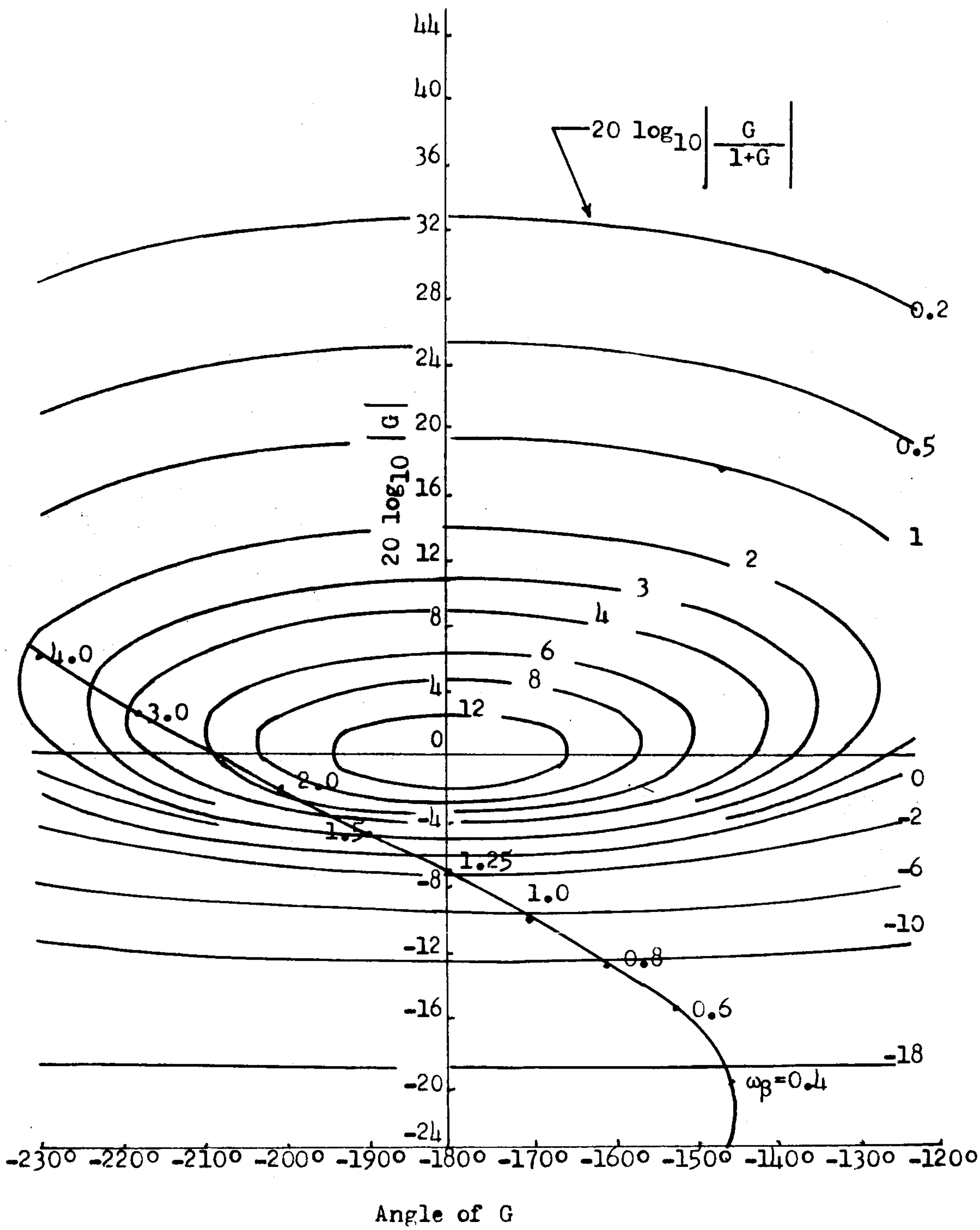


FIG. 25. - Gain-phase plot for $G(s) = \dfrac{K}{s(s+1)}$.

The Root Locus Method

The root locus technique has become one of the most popular methods available for analysis and design of linear control systems. It has been shown previously that the location of the poles and zeros of the closed-loop transfer function $C(x_p) / U(x_p)$ completely determine the stability and transient response of each sampled-data system. Furthermore, the closed-loop x_p-transfer function is a rational function of x_p and the characteristic equation of the system shown in Figure 19 is an algebraic equation in x_p. This characteristic equation is

$$1 + G_R(x_p) = 0 \tag{IV-46}$$

The construction of the x_p-plane root locus is straightforward since the same rules which apply for the conventional root locus diagrams are also applicable to the x_p-plane loci. The rules of construction are based on the following conditions:

(1) $$\left| G_R(x_p) \right| = 1 \tag{IV-47}$$

(2) $$\underline{/ G_R(x_p)} = 180^\circ + k(360^\circ) \tag{IV-48}$$

where $k = 0, \pm 1, \pm 2, \ldots$ all integers.

The rules and their proofs may be found in most standard textbooks on feedback control systems.[16]

As an example, consider the same system used in the previous stability study methods. This system is redrawn in Figure 27. Since

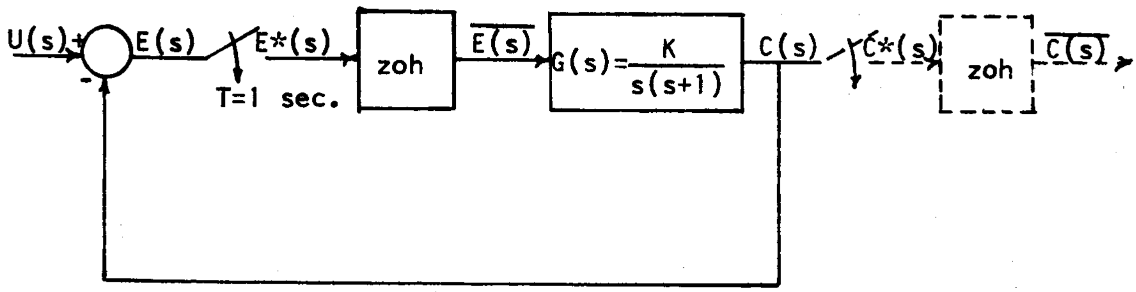


Fig. 27. - Closed-loop sampled-data system.

the system has unity feedback, H(s) = 1, the characteristic equation is

$$1 + G_R(x_p) = 0 \qquad \text{(IV-49)}$$

The open-loop x_p-transfer function is

$$G_R(x_p) = K\left[\frac{0.264\, x_p^2 + 0.368\, x_p}{(1 - x_p)(1 - 0.368\, x_p)}\right] \qquad \text{(IV-50)}$$

for a sampling period T of one second. The root loci of the system are plots of the roots of (IV-49) when K is varied from zero to infinity (K may also be negative). Equation (IV-50) has poles at $x_p = 1$ and $x_p = 2.72$ and zeros at $x_p = 0$ and $x_p = -1.39$. From the rules for root locus construction, the root loci plots may be drawn as shown in Figure 28.

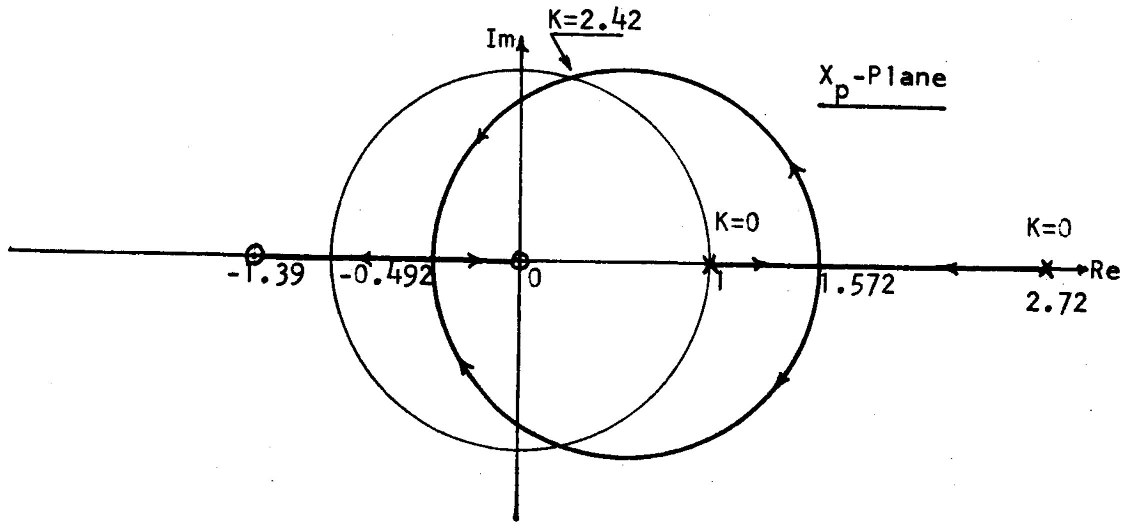


Fig. 28. - Root locus plot of the sampled-data system shown in Fig. 21.

The loci start from the poles $x_p = 1$ and $x_p = 2.72$ and terminate on the zeros $x_p = -1.39$ and $x_p = 0$. The breakaway points are located by the following procedure:

(1) Write the characteristic equation as

$$K = f(x_p) \tag{IV-51a}$$

(2) The breakaway points are the roots of

$$\frac{dK}{d\,x_p} = 0 \tag{IV-51b}$$

Therefore, the breakaway points for this example are at $x_p = 1.512$ and $x_p = -0.472$. The marginal gain, K_c, may be determined graphically or from (IV-49). Rearranging (IV-49) gives

$$K' = \frac{|1 - x_p| \quad |2.72 - x_p|}{|x_p| \quad |x_p + 1.39|} \tag{IV-52}$$

where

$$K' = 0.717\ K \tag{IV-53}$$

The critical value of gain is that value of K at the point where the locus enters the unit circle. From Figure 28 the magnitudes of (IV-52) may be determined giving

$$K' = \frac{(2.7)(1.2)}{(1)(1.87)} = 1.73 \tag{IV-54}$$

Therefore, the critical value of gain is

$$K_c = \frac{K'}{0.717} = \frac{1.73}{0.717} = 2.42 \tag{IV-55}$$

It is obvious from Figure 28 that the lower limit on K for a stable system is K = 0. Therefore, for a stable system

$$0 < K < 2.42 \tag{IV-56}$$

The gain at any other point on the locus may be determined in a similar manner.

An interesting point is seen from Figure 28. For a two pole, two zero configuration the complex conjugate section of the root loci is a circle. This may be proved as follows:

$$\text{Let } x_p = x + jy \tag{IV-57}$$

Then (IV-51) becomes

$$G_R(x_p) = \frac{K\left[0.264\ (x + jy)^2 + 0.368\ (x + jy)\right]}{(1 - x - jy)(1 - 0.368\ x - j\ 0.368\ y)} \tag{IV-58}$$

Condition (2), equation (IV-49), specified earlier for root loci construction, states that

$$\angle G_R(x_p) = 180^\circ + k(360^\circ) = (2k + 1)\pi \tag{IV-59}$$

Equation (IV-58) may be written partially in view of (IV-59) as

$$\angle G_R(x_p) = \sum_{i=1}^{m} \angle x_p - z_i - \sum_{j=1}^{m+n} \angle x_p - p_j \tag{IV-60}$$

where z_i indicates zeros and p_j indicates poles of $G_R(x_p)$. Therefore,

$$\angle G_R(x_p) = \tan^{-1}\left[\frac{0.528\ x\ y + 0.368\ y}{0.264\ x^2 - 0.264\ y^2 + 0.368\ x}\right]$$

$$- \tan^{-1}\left[\frac{0.736\ x\ y - 1.368\ y}{1 - 1.368\ x + 0.368\ x^2 - 0.368\ y^2}\right] \tag{IV-61}$$

Taking the tangent of both sides and using the identity

$$\tan\,(a + b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} \qquad \text{(IV-62)}$$

give

$$\tan \angle G_R(x_p) = \frac{\left[\dfrac{0.528\,x\,y + 0.368\,y}{0.264\,x^2 - 0.264\,y^2 + 0.368\,x} - \dfrac{0.736\,x\,y - 1.368\,y}{1 - 1.368\,x + 0.368\,x^2 - 0.368\,y^2}\right]}{1 + \left[\dfrac{0.528\,x\,y + 0.368\,y}{0.264\,x^2 - 0.264\,y^2 + 0.368\,x}\right] \cdot \left[\dfrac{0.736\,x\,y - 1.368\,y}{1 - 1.368\,x + 0.368\,x^2 - 0.368\,y^2}\right]} \qquad \text{(IV-63)}$$

Now

$$\tan \angle G_R(x_p) = \tan\,(2\,k + 1)\pi = 0 \qquad \text{(IV-64)}$$

Simplifying (IV-63) by (IV-64) gives

$$(x - 0.543)^2 + 1.0\ y^2 = 1.298 \qquad \text{(IV-65)}$$

This is the equation of a circle with center at (0.543, 0) in the x_p-plane and a radius r = 1.14.

From the preceding example it is seen that the root locus plot in the x_p-plane for a sampled-data-hold system may be made just as in the continuous case. Because of a different region of stability, a different interpretation must be placed on the resultant diagram. The stable region is the area outside the unit circle.

V. RESPONSE OF SAMPLED-DATA SYSTEMS BETWEEN SAMPLING INSTANTS

The x-transform analysis has been developed around a combination circuit including an ideal sampler and a zero-order hold. The output of such a combination approximates the unsampled output of the system closely if the sampling rate is sufficiently high. The output of a sampled-data-hold system is equal to the unsampled system output only at the sampling instants; it is an approximation between sampling instants. In order to get a complete description of the system, it is necessary to know something about its behavior not only at the sampling instants but also between sampling instants. The two methods that will be investigated for determining the system behavior between sampling instants are the submultiple sampling method and the modified x-transform method.

Submultiple Sampling

Open-loop

One approach to the evaluation of system outputs between sampling instants may involve the use of a fictitious sampler-hold at the output of the system whose period is a fraction of that at the input.[17] The general theory considers the case where the output sampler is operated at a period T/n, n being an integer, and the input sampler is operated at a period T.

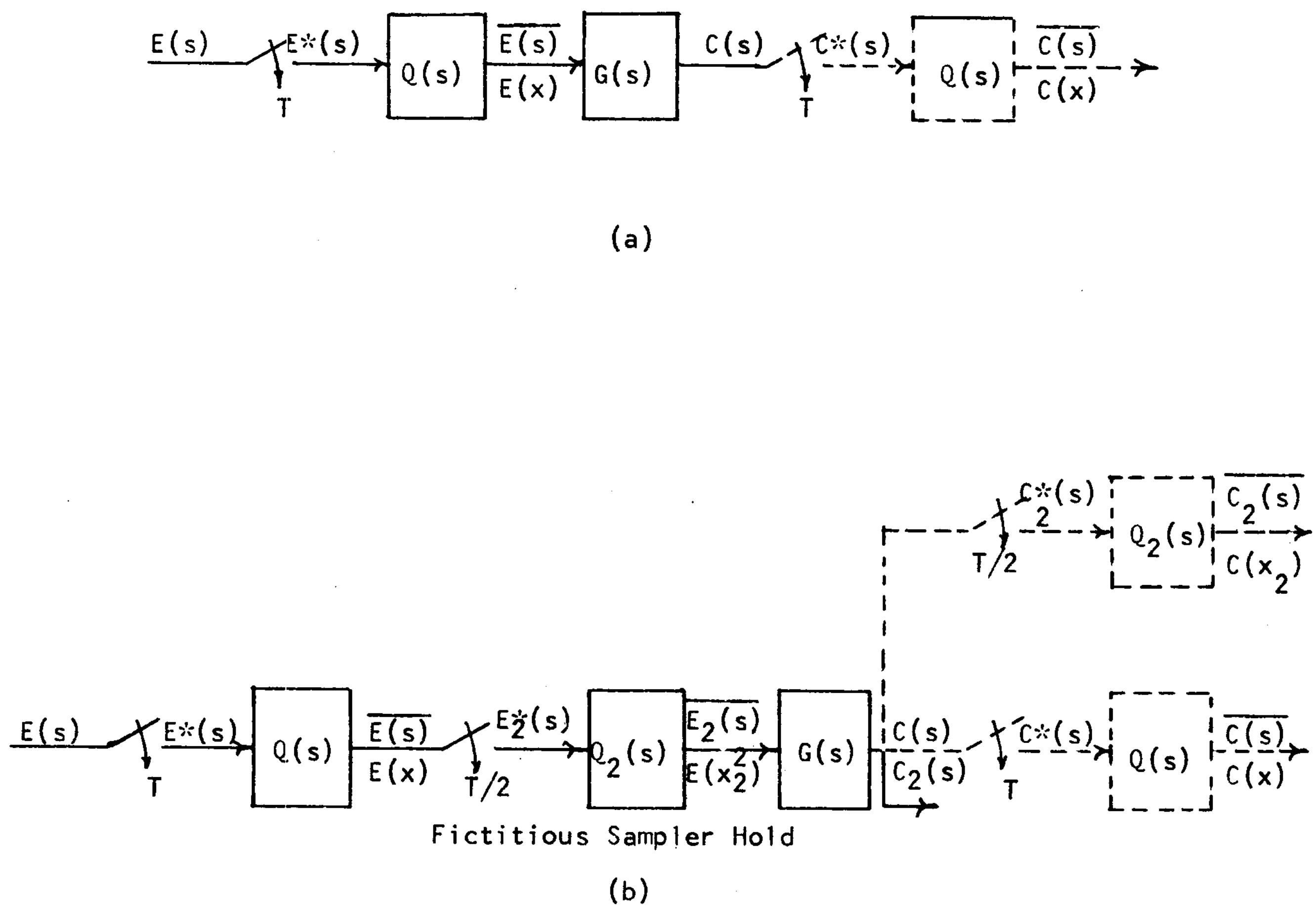


Fig. 29. - (a) Open-loop sampled-data system. (b) Open-loop sampled-data system with fictitious samplers for determining the ouptut between samlling instants.

In order to gain a basic understanding of the principles of submultiple sampling, the specific case of a double-rate output sampler will be considered. The system shown in Figure 29 is to be investigated.

Since $\overline{E(s)}$ is constant between sampling instants, the presence of the fictitious sampler in the input does not affect the original system. At alternate sampling instants the input samplers operate simultaneously. Between these sampling instants the fictitious sampler merely samples $\overline{E(s)}$ and holds it until the two samplers again operate simultaneously (see Figure 30).

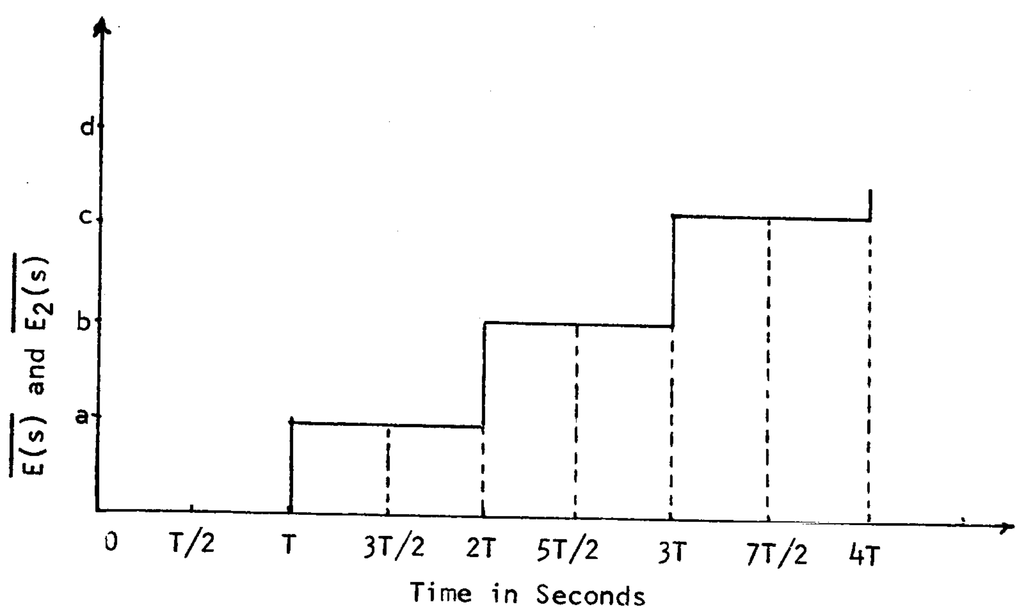


Fig. 30. - Results of a sampler with period T followed by a sampler with period T/2.

While the use of the double frequency sampler on the input does not alter the input to G(s), it does suggest that a change of variable might be useful. This new variable is defined as

$$x_2 \triangleq \frac{e^{-(T/2)s}}{s} \qquad \text{(V-1)}$$

From (V-1) it is seen that

$$x = x_2 \, x_{2p} \qquad \text{(V-2)}$$

However, once again, it is not necessary to differentiate between x and x_p, and (V-2) may be expressed as

$$x = x_2^2 \qquad \text{(V-3)}$$

Therefore, in view of (V-3), it is seen that the input to G(s) is

$$\mathcal{X}\left[\overline{E(s)}\right] = E(x_2^2) \qquad \text{(V-4)}$$

The input can now be expressed in terms of the double-rate variable x_2 by merely replacing all the x's by x_2^2.

The double-rate output x-transform is related to the input double-rate transform by

$$C(x_2) = G_R(x_2) \; E(x_2^2) \qquad \text{(V-5)}$$

where

$$G_R(x_2) \triangleq \mathcal{X}\left[\frac{G(s)}{s}\right]_{\substack{x \to x_2 \\ T \to T/2}} \qquad \text{(V-6)}$$

Equation (V-5) may be developed from the block diagram of Figure 29b as follows:

$$\overline{C_2(s)} = Q_2(s)\ C_2^*(s) \qquad \text{(V-7)}$$

$$R_2(s) \triangleq Q_2(s)\ G(s) \qquad \text{(V-8)}$$

and

$$C_2(s) = G(s)\ Q_2(s)\ E_2^*(s) \qquad \text{(V-9)}$$

Starring (V-9) after substituting (V-8) gives

$$C_2^*(s) = R_2^*(s)\ E_2^*(s) \qquad \text{(V-10)}$$

where the subscript indicates that the sampling period is T/2. Substituting (V-10) into (V-7) yields

$$\overline{C_2(s)} = Q_2(s)\ E_2^*(s)\ R_2^*(s)$$

$$= \overline{E_2(s)}\ R_2^*(s) \qquad \text{(V-11)}$$

However, the input sampler operating at a period T/2 does not affect the input to G(s) and (V-11) may be written as

$$\overline{C_2(s)} = \overline{E(s)}\, R_2^*(s) \qquad \text{(V-12)}$$

Under the definitions given in Chapter III for the open-loop transfer function and equation (V-4), one may write (V-12) in the x-transform notation as

$$C(x_2) = E(x_2^2)\, G_R(x_2) \qquad \text{(V-13)}$$

As an example of the application of the submultiple sampling method for determining the output between sampling instants, consider the system of Figure 29b with $G(s) = 1/(s + 1)$ and $E(s) = 1/s$. Determine the output at $t = 0, 0.5, 1.0, 1.5, 2 \ldots$ seconds when $T = 1$ second. The solution is as follows:

$$G_R(x) = \chi\left[\frac{G(s)}{s}\right] = \chi\left[\frac{1}{s(s+1)}\right] = \frac{x - e^{-T}x}{1 - e^{-T}x} \qquad \text{(V-14)}$$

and

$$G_R(x_2) = \frac{x_2 - e^{-T/2}x_2}{1 - e^{-T/2}x_2} \qquad \text{(V-15)}$$

The input is

$$E(x) = \mathcal{X}\left[\frac{1}{s}\right] = x^0 \tag{V-16}$$

From (V-2)

$$x^0 = x_2^0 \, x_{2p}^0 \tag{V-17}$$

But $x_{2p}^0 = 1$. Then

$$E(x_2^2) = x_2^0 \tag{V-18}$$

Substituting (V-15) and (V-18) into (V-13) gives

$$C(x_2) = x_2^0 \left[\frac{0.394\ x_2}{1 - 0.606\ x_2}\right] \tag{V-19}$$

Taking the inverse x-transform gives

$$C(x_2) = 0.394\ x_2 + 0.239\ x_2^2 + 0.145\ x_2^3$$

$$+ 0.088\ x_2^4 + 0.0534 x_2^5 + \ldots \tag{V-20}$$

$\overline{c_2(t)}$ is obtained from (V-20) and plotted in Figure 31.

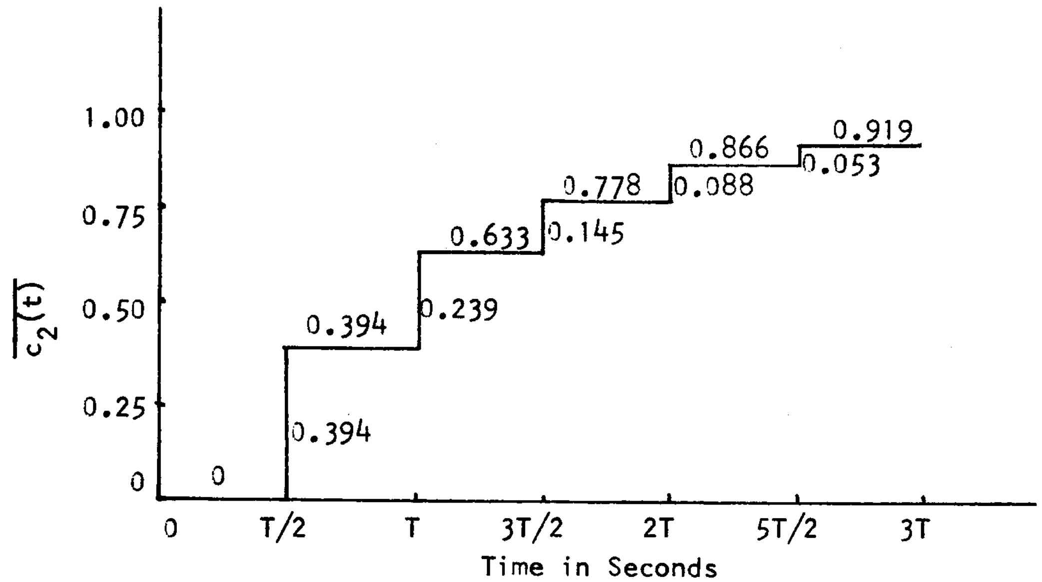


Fig. 31. - Submultiple sampling with $T \rightarrow T/2$ and $x \rightarrow x_2^2$.

It is apparent from the double-rate case previously considered that an n-rate case may be given. In general, the submultiple rate sampling method for the open-loop system is determined by

$$C(x_n) = E(x_n^n)\, G_R(x_n) \tag{V-21}$$

where

$$G_R(x_n) = G_R(x) \Bigg|_{\substack{x \rightarrow x_n \\ T \rightarrow T/n}} \tag{V-22}$$

and

$$E(x_n^n) = E(x) \Big|_{x \to x_n^n} \qquad \text{(V-23)}$$

The number of additional sampled values, q, desired between any two consecutive sampling instants is determined by $n = q + 1$.

Closed-loop

The application of the submultiple sampling method for evaluating the response between sampling instants of a closed-loop system is similar to that for an open-loop system. Figure 32 shows a closed-loop system with fictitious sample-hold circuits included. The sampler on the input operating with a period of T/n seconds does not alter the input to G(s) (see the open-loop analysis on page 90).

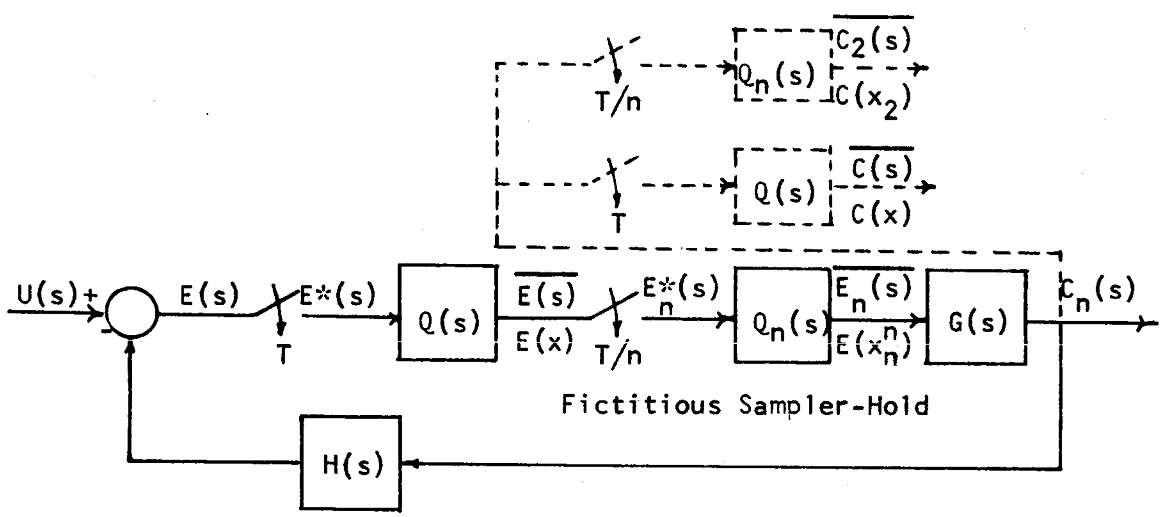


Fig. 32. - Closed-loop sampled-data system showing fictitious samplers which aid in determining the response between sampling instants.

The fictitious sampler at the output gives

$$C(x_n) = E(x_n^n)\ G_R(x_n) \qquad \text{(V-24)}$$

where $E(x_n^n)$ is the ordinary x-transform with x replaced by x_n^n and $G_R(x_n)$ is the ordinary x-transform for a transfer-function with x replaced by x_n and T replaced by T/n.

The output of the fictitious sampler-hold on the input is

$$E(x_n^n) = U(x_n^n) - E(x_n^n)\ (GH)_R(x_n^n) \qquad \text{(V-25)}$$

Solving for $E(x_n^n)$,

$$E(x_n^n) = \frac{U(x_n^n)}{1 + (GH)_R\ (x_n^n)} \qquad \text{(V-26)}$$

Substituting (V-26) into (V-24) gives

$$C(x_n) = G_R(x_n) \frac{U(x_n^n)}{1 + (GH)_R\ (x_n^n)} \qquad \text{(V-27)}$$

where

$$(GH)_R\ (x_n^n) = (GH)_R\ (x) \Big|_{x \longrightarrow x_n^n} \qquad \text{(V-28)}$$

and

$$(GH)_R\ (x) = \mathcal{Z}\left[\frac{H(s)\ G(s)}{s}\right] \tag{V-29}$$

Consider the system of Figure 32 with $G(s) = 1/s(s + 1)$, $H(s) = 1$ and $T = 1$ second. It is desired to find the output response at two additional instants (equally spaced) during each sampling period if the input is a unit step function. The solution is as follows:

$$G_R(x) = \mathcal{Z}\left[\frac{G(s)}{s}\right] = \mathcal{Z}\left[\frac{1}{s^2\ (s + 1)}\right]$$

$$= \frac{x\ (T - e^{-T} - 1) + x^2\ (1 - e^{-T}\ T - e^{-T})}{1 - (1 + e^{-T})\ x + e^{-T}\ x^2} \tag{V-30}$$

Determining $G_R(x_3)$ from (V-30) gives

$$G_R(x_3) = \frac{\left[x_3\ (0.333 + e^{-0.333} - 1) + x_3^2\ (1 - 0.333\ e^{-0.333} - e^{-0.333})\right]}{1 - (1 + e^{-0.333})\ x_3 + e^{-0.333}\ x_3^2}$$

or

$$G_R(x_3) = \frac{0.05\ x_3 + 0.44\ x_3^2}{1 - 1.717\ x_3 + 0.717\ x_3^2} \tag{V-31}$$

The input is a unit step; therefore

$$U(x) = x^0$$

and

$$U(x_3^3) = x_3^0 \tag{V-32}$$

Using (V-30)

$$(GH)_R\ (x_3^3) = G_R(x_3^3) = \frac{0.264\ x_3^6 + 0.368\ x_3^3}{0.368\ x_3^6 - 1.368\ x_3^3 + 1} \tag{V-33}$$

Substituting (V-31), (V-32) and (V-33) into (V-27) gives

$$C(x_3) = \frac{\left[\dfrac{0.05\ x_3 + 0.044\ x_3^2}{1 - 1.717\ x_3 + 0.717\ x_3^2}\right] x_3^0}{1 + \left[\dfrac{0.264\ x_3^6 + 0.368\ x_3^3}{0.368\ x_3^6 - 1.368\ x_3^3 + 1}\right]} \tag{V-34}$$

Simplifying (V-34) yields

$$C(x_3) = 0.050\ x_3 + 0.130\ x_3^2 + 0.187\ x_3^3 + 0.209\ x_3^4 + 0.207\ x_3^5 + 0.208\ x_3^6 + \dots \qquad \text{(V-35)}$$

The coefficients of x_3^3, x_3^6, x_3^9, ... which correspond to the output at t = T, 2T, 3T, ... may be checked with the results for the output of this same system in Chapter III, page 45.

The Delayed X-Transform

The result of a delay factor in the x-transform theory was shown in Chapter II, page 24. This delay factor or shift is

$$\mathcal{X}\left[e(t - nT)\ u(t - nT)\right] = x^n\ E(x) \qquad \text{(V-36)}$$

where n is an integer. Suppose that the delay is kT where k is not an integer and suppose that the sampling instant remains unchanged; that is, the sampling and hold occur at t = 0, T, 2T, ... , but the delay is not necessarily an integral number of sampling periods. Figure 33 illustrates this condition. The sampled-hold delay sequence may be represented by

$$\overline{e(t - kT)} = \sum_{n=0}^{\infty} e(nT - kT) \left[u(t - nT) - u\left[t - (n + 1)\ T\right]\right] \qquad \text{(V-37)}$$

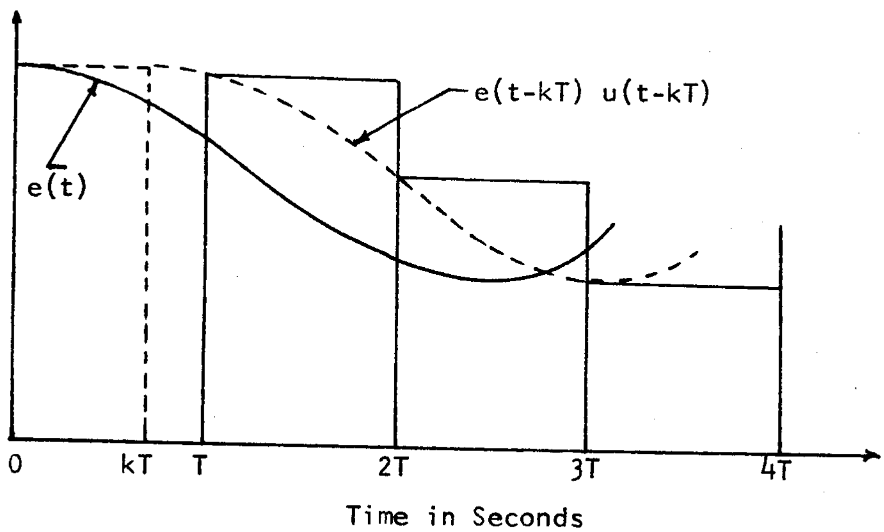


Fig. 33. - Delay function and pulse sequence of delay function.

where k is a noninteger and n = 0, 1, 2, 3, ... It is assumed that k can be represented as the sum of two quantities. Let

$$k = p + q \tag{V-38}$$

where p is the largest integer less than k and q is a positive number less than one. Equation (V-37) becomes

$$\overline{e\ (t - kT)} =$$

$$\sum_{n=0}^{\infty} e(nT - pT - qT) \Big[u(t - nT) - u \big[t - (n + 1)\ T \big] \Big] \tag{V-39}$$

Taking the x-transform of both sides of (V-39) gives

$$\mathcal{X}\left[\overline{e(t - kT)}\right] = x^p \sum_{n=0}^{\infty} e(nT - qT)(x^n - x^{n+1}) \qquad \text{(V-40)}$$

Since there is no additional information gained by delaying an integral number of sampling periods, p is set equal to zero. Equation (V-40) then becomes

$$\mathcal{X}\left[e(t - qT\ u(t - qT)\right] = \sum_{n=0}^{\infty} e(nT - qT)(x^n - x^{n+1}) \qquad \text{(V-41)}$$

Equation (V-41) will be referred to as the defining equation for the delayed x-transform of e(t) and denoted by

$$E(x, q) \triangleq \mathcal{X}\left[e(t - qT)\ u(t - qT)\right] \qquad \text{(V-42)}$$

If $q = 0$, then there is no shift and the summation over n in (V-41) is from zero to infinity; however, if q is other than zero, then the summation is from one to infinity.

As an example, it is desired to determine the delayed x-transform for a time function e^{-at} that is delayed by 0.5 T seconds. Using (V-41)

$$\begin{aligned} E(x, q = 0.5) &= \mathcal{X}\left[e(t - 0.5T)\ u(t - 0.5T)\right] \\ &= \sum_{n=0}^{\infty} e^{-a(nT - 0.5T)} (x^n - x^{n+1}) \end{aligned} \qquad \text{(V-43)}$$

It should be noted that the function is zero until $t = 0.5T$; therefore, the summation should be from $n = 1$ to infinity. Thus,

$$E(x, 0.5) = \sum_{n=1}^{\infty} e^{-a(nT - 0.5T)} (x^n - x^{n+1})$$

$$= (x^0 - x)\, e^{-0.5aT} (x + x^2 e^{-aT} + x^3 e^{-2aT} + \dots)$$

$$= \frac{(x - x^2)\, e^{-0.5aT}}{1 - e^{-aT} x} \tag{V-44}$$

The inverse of (V-44) is plotted in Figure 34 along with the continuous function for comparison.

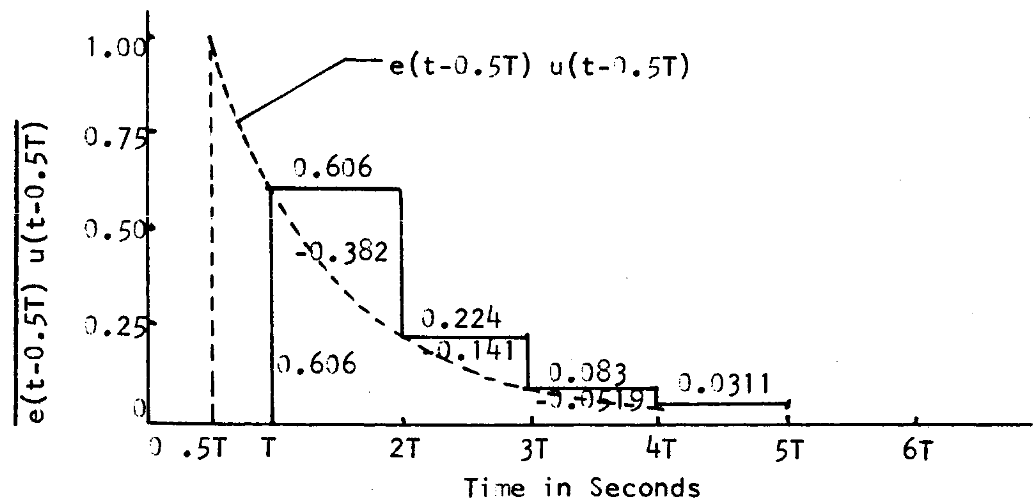


Fig. 34. - Comparison $e(t) = e^{-at}$ delayed by 0.5T and $\chi\left[e(t - 0.5T)\, u(t - 0.5T)\right]$.

The Modified X-Transform

Open-loop

The system shown in Figure 35 is an open-loop system with a

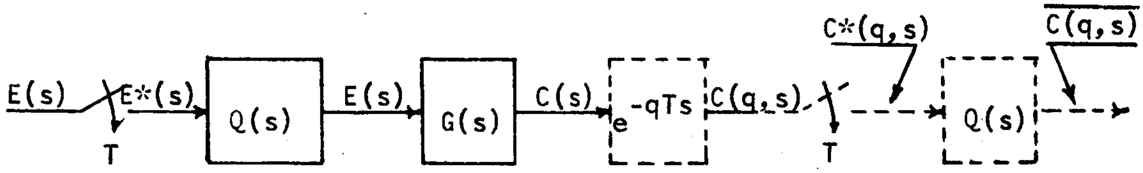


Fig. 35. - Open-loop system with fictitious time delay e^{-qTs}.

fictitious time delay inserted at the output. The fictitious time delay is not a part of the system; it is added merely as a convenience for allowing the determination of the output response at times other that the sampling instants. The definition of q is the same as in the delayed x-transform analysis.

It is assumed that the fictitious delay is grouped with G(s) and the following definition is made:

$$G'(s) \triangleq G(s)\, e^{-qTs} \qquad \text{(V-45)}$$

By the same block diagram manipulations as are shown in Chapter III for the open-loop case, it can be shown that

$$\overline{C(s,\ q)} = \overline{E(s)}\, R^{*}(s,\ q) \qquad \text{(V-46)}$$

or

$$C(x, q) = E(x)\, G_R(x, q) \tag{V-47}$$

where

$$G_R(x, q) \triangleq \mathcal{L}\left[\frac{G'(s)}{s}\right] \tag{V-48}$$

At this point a change in the delay nomenclature may be made. Let

$$q = 1 - m \tag{V-49}$$

Then

$$C(x, m) = E(x)\, G_R(x, q)\Big|_{q = 1 - m} \tag{V-50}$$

Since q is a number between zero and one, m is also between zero and one. Therefore, using the delay theorem

$$G_R(x, m) = G_R(x, q)\Big|_{q = 1 - m}$$

$$= x \sum_{n=0}^{\infty} g_R\left[(n + m)\, T\right] \left(x^n - x^{n+1}\right) \tag{V-51}$$

where the rule for taking the x-transform of a transfer function carries over to the modified x-transform. The modified x-transform, $E(x, m)$, is given by

$$\mathcal{X}_m \left[e(t)\right] \triangleq E(x, m) = x \sum_{n=0}^{\infty} e\left[(n + m) T\right] \cdot (x^n - x^{n+1}) \qquad \text{(V-52)}$$

It should be noted that the modified x-transform involves a delay and therefore there will never be an x^0 term in the output response of $C(x, m)$. An example will illustrate the modified x-transform method. It is desired to determine the output response of the system of Figure 35 at $t = 0.5, 1.5, 2.5, \ldots$ with $G(s) = 1/(s + 1)$ and a unit step input. The solution is as follows:

Using (V-51), the transfer-function transform is

$$G_R(x, m) = \mathcal{X}_m \left[\frac{G(s)}{s}\right]$$

$$= \mathcal{X}_m \left[\frac{1}{s(s + 1)}\right]$$

$$= \mathcal{X}_m \left[u(t) - e^{-t}\right] \qquad \text{(V-53)}$$

which from (V-52) becomes

$$G_R(x, m) = x - \frac{(x - x^2)\, e^{-mT}}{1 - e^{-T} x} \qquad \text{(V-54)}$$

or

$$G_R(x, m) = \frac{x(1 - e^{-mT}) + x^2 (e^{-mT} - e^{-T})}{1 - e^{-T} x} \qquad \text{(V-55)}$$

In order to get the output response at t = 0.5, 1.5, 2.5, ... , (V-50) will be used with m set equal to 0.5. The input is a unit step, therefore $\mathcal{X}\left[u(t)\right] = x^0$ and T is assumed to be one. Then

$$C(x, 0.5) = x^0 \, G_R(x, 0.5)$$

$$= \frac{0.394\, x + 0.238\, x^2}{1 - 0.368\, x}$$

$$= 0.394\, x + 0.383\, x^2 + 0.141\, x^3 + \ldots \qquad \text{(V-56)}$$

$\overline{c(t, 0.5)}$ is obtained from (V-56) and plotted in Figure 36 along with $\overline{c(t, 1)}$ and $\overline{c(t, 0)}$. It should be noted that

$$C(x, 1) = C(x) \qquad \left[\text{if } e(0) = 0\right] \qquad \text{(V-57)}$$

and

$$C(x, 0) = x\, C(x) \qquad \text{(V-58)}$$

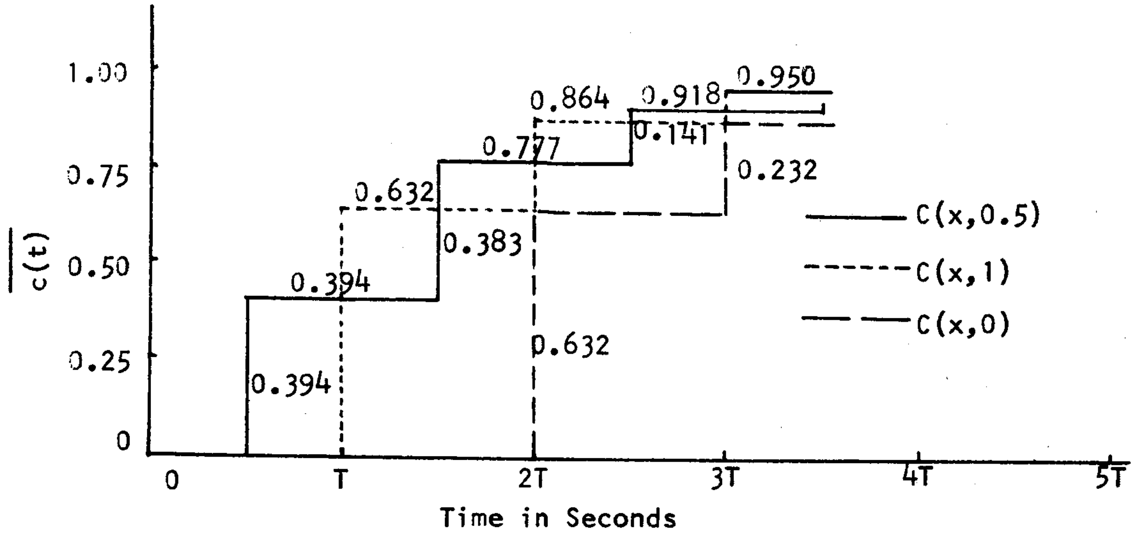


Fig. 36. - C(x, m) for step input and G(s) = 1/(s + 1), T = 1.

A modified z-transform analysis for this same system gives

$$C(z,\ 0.5) \;=\; 0.394\ z^{-1},\ +\ 0.777\ z^{-2} + 0.918\ z^{-3} + \ldots \qquad \text{(V-59)}$$

These values are also indicated in Figure 36.

Evaluation of some modified x-transforms

Determination of the modified x-transform of the time function u(t). - By definition

$$\mathcal{X}_m\left[u(t)\right] \;\overset{\Delta}{=}\; E(x,\ m)$$

$$= x \sum_{n=0}^{\infty} e\left[(n+m)T\right]\ (x^n - x^{n+1})$$

$$= x \sum_{n=0}^{\infty} u\left[(n+m)T\right]\ (x^n - x^{n+1})$$

$$\mathcal{X}_m\left[u(t)\right] = x\,(x^0 - x + x - x^2 + x^2 - \dots)$$

$$= x \qquad \text{(V-60)}$$

<u>Determination of the modified x-transform of the time function e^{-at}.</u> Using (V-52) gives

$$\mathcal{X}_m\left[e^{-at}\right] = x\sum_{n=0}^{\infty} e^{-a(n+m)T}\,(x^n - x^{n+1})$$

$$= x\,e^{-amT}\,(x^0 - x)(1 + e^{-aT}x + e^{-2aT} + x^2 + \dots)$$

$$\mathcal{X}_m\left[e^{-at}\right] = \frac{e^{-amT}\,(x - x^2)}{1 - e^{-aT}x} \qquad \text{(V-61)}$$

<u>Determination of the modified x-transform of the time function $(1/a)\,[t - (1 - e^{-at})/a]$.</u> - Taking the transform of the sum gives

$$\mathcal{X}_m\left[\frac{1}{a}\left(t - \frac{1 - e^{-at}}{a}\right)\right]$$

$$= \mathcal{X}_m\left[\frac{t}{a}\right] - \mathcal{X}_m\left[\frac{1}{a^2}\,u(t)\right] + \mathcal{X}_m\left[\frac{e^{-at}}{a^2}\right]$$

$$= \frac{1}{a}\left[mTx + \frac{x^2\,T}{1 - x}\right] - \frac{1}{a^2}\left[x\right] + \frac{1}{a^2}\left[\frac{e^{-amT}\,(x - x^2)}{1 - e^{-aT}x}\right]$$

$$\mathcal{X}_m \left[\frac{1}{a} \left(t - \frac{1 - e^{-at}}{a} \right) \right]$$

$$= \frac{1}{a} \left[\frac{mT\,(x - x^2) + Tx^2}{1 - x} - \frac{x}{a} + \frac{e^{-amT}\,(x - x^2)}{a\,(1 - e^{-aT}\,x)} \right] \qquad \text{(V-62)}$$

Closed-loop

A modified x-transform analysis of a closed-loop system may be made using the system shown in Figure 37. From this block diagram

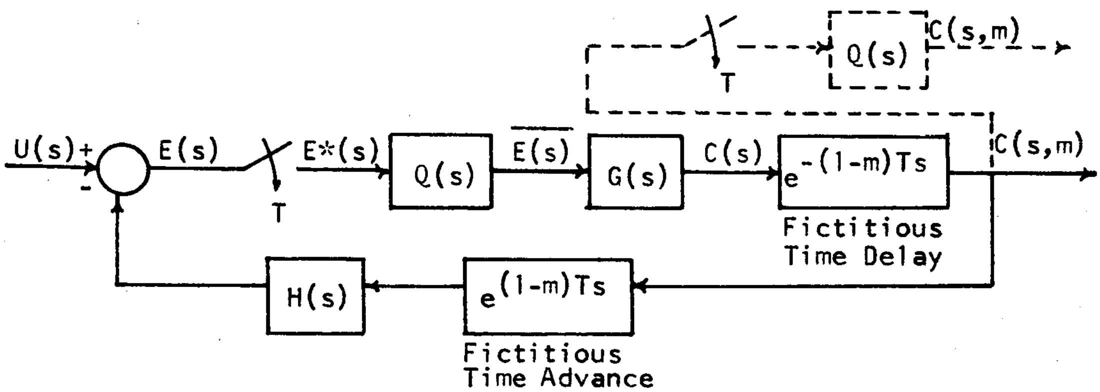


Fig. 37. - Closed-loop sampled-data system with fictitious time delay and advance.

the following equations may be written:

$$C(s,\ m) = G'(s)\ \overline{E(s)} \qquad \text{(V-63)}$$

where

$$G'(s) = G(s)\ e^{-(1\ -\ m)Ts} \qquad \text{(V-64)}$$

The error signal is

$$E(s) = U(s) - H'(s)\ C(s,\ m) \tag{V-65}$$

where

$$H'(s) = H(s)\ e^{(1\ -\ m)Ts} \tag{V-66}$$

Substituting (V-63) into (V-65) gives

$$E(s) = U(s) - H'(s)\ G'(s)\ \overline{E(s)} \tag{V-67}$$

Taking the barred transform of both sides gives

$$\overline{E(s)} = \overline{U(s)} - \left[\frac{\overline{H'(s)\ G'(s)}}{s}\right] \overline{E(s)} \tag{V-68}$$

Solving for $\overline{E(s)}$ gives

$$\overline{E(s)} = \frac{\overline{U(s)}}{1 + \left[\dfrac{\overline{H'(s)\ G'(s)}}{s}\right]} \tag{V-69}$$

Taking the barred transform of (V-63) and substituting (V-69) into it yield

$$\overline{C(s,\ m)} = \frac{\left[\dfrac{\overline{G'(s)}}{s}\right] \overline{U(s)}}{1 + \left[\dfrac{\overline{H'(s)\ G'(s)}}{s}\right]} \qquad \text{(V-70)}$$

or

$$C(x,\ m) = \frac{G_R(x,\ m)\ U(x)}{1 + (GH)_R\ (x)} \qquad \text{(V-71)}$$

The effects of time delay and time advance cancel each other in the loop gain portion of (V-70). By a previous definition

$$(GH)_R(x) = \chi\left[\frac{H(s)\ G(s)}{s}\right] \qquad \text{(V-72)}$$

For a system with unity feedback

$$C(x,\ m) = \frac{G_R(x,\ m)\ U(x)}{1 + G_R(x)} \qquad \text{(V-73)}$$

<u>Modified x-transform theorems</u>

<u>Shifting theorem.</u> - If $\chi_m\left[e(t)\right] = E(x,\ m)$, then

$$\mathcal{X}_m\left[e(t - nT)\, u(t - nT)\right] = x^n\, E(x, m) \tag{V-74}$$

The proof of this theorem is given in the derivation of the modified x-transform through the use of the delayed transform.

<u>Corollary I.</u> If $\mathcal{X}_m\left[e(t)\right] = E(x, m)$, then

$$\mathcal{X}_m\left[e(t + T)\right] = x^{-1}\, E(x, m) - e(mT)(x^0 - x) \tag{V-75}$$

<u>Proof</u>: By definition

$$\mathcal{X}_m\left[e(t + T)\right] = x \sum_{n=0}^{\infty} e\left[(n + 1 + m)\, T\right] (x^n - x^{n+1})$$

$$= \sum_{n=0}^{\infty} e\left[(n + 1 + m)T\right] (x^{n+1} - x^{n+2}) \tag{V-76}$$

Let $k = n + 1$. Therefore,

$$\mathcal{X}_m\left[e(t + T)\right] = \sum_{k=1}^{\infty} e\left[(k + m)\, T\right] (x^k - x^{k+1}) \tag{V-77}$$

Adding and subtracting the term $e(mT)(x^0 - x)$ under the summation sign gives

$$\mathcal{X}_m\left[e(t+T)\right] = \sum_{k=0}^{\infty} e\left[(k+m)T\right](x^k - x^{k+1}) - e(mT)(x^0 - x)$$

$$= x^{-1}E(x, m) - e(mT)(x^0 - x) \qquad \text{Q.E.D.} \quad \text{(V-78)}$$

For illustrations see Examples 8 and 9, Appendix B.

Initial value theorem. - If $\mathcal{X}_m\left[e(t)\right] = E(x, m)$, then

$$\lim_{t\to 0^+} e(t) = \lim_{\substack{n\to 0\\ m=0}} e(nT, m) = \lim_{\substack{x\to 0\\ m=0}} \left[\frac{E(x, m)}{x(x^0 - x)}\right]_{x^0 = 1} \qquad \text{(V-79)}$$

Proof: By definition

$$E(x, m) = x\sum_{n=0}^{\infty} e\left[(n+m)T\right](x^n - x^{n+1})$$

or

$$x^{-1}E(x, m) = \sum_{n=0}^{\infty} e\left[(n+m)T\right](x^n - x^{n+1}) \qquad \text{(V-80)}$$

Letting m equal zero in (V-80) gives

$$x^{-1}E(x, m)\Big|_{m=0} = \sum_{n=0}^{\infty} e(nT)(x^n - x^{n+1}) \qquad \text{(V-81)}$$

The right side of (V-81) is recognized as the ordinary x-transform of e(t). Therefore, by the application of the initial value theorem for the ordinary x-transform (V-81) becomes

$$\lim_{\substack{m=0 \\ x \to 0}} \left[\frac{E(x, m)}{x(x^0 - x)} \right] = \lim_{x \to 0} \left[\frac{E(x)}{x^0 - x} \right]$$

$$= \lim_{t \to 0} e(t) \quad \text{Q.E.D.} \qquad \text{(V-82)}$$

An example of the use of this theorem is given in Example 10, Appendix B.

<u>Final value theorem.</u> - If $\mathcal{X}_m \left[e(t)\right] = E(x, m)$, then,

$$\lim_{\substack{n \to \infty \\ 0 \leq m \leq 1}} e(nT, m) = \lim_{\substack{x \to 1 \\ 0 \leq m \leq 1}} E(x, m)\Big|_{x^0 = 1} \qquad \text{(V-83)}$$

<u>Proof</u>: From the shifting theorem

$$\mathcal{X}_m \left[e(t + T)\right] = x_p^{-1} E(x, m) - e(mT)(x^0 - x) \qquad \text{(V-84)}$$

Then

$$\mathcal{X}_m \left[e(t + T) - e(t) \right] = x_p^{-1} E(x, m) - e(mT)(x^0 - x)$$

$$- E(x, m) \qquad \text{(V-85)}$$

Rearranging (V-85) gives

$$\frac{\mathcal{X}_m \left[e(t + T) - e(t) \right]}{1 - x_p} = \frac{E(x, m)}{x_p} - e(mT)\, x^0 \qquad \text{(V-86)}$$

However, by definition

$$\mathcal{X}_m \left[e(t + T) - e(t) \right] = \lim_{k \to \infty} x_p \sum_{n=0}^{k} \left[e\left[(n + 1 + m)T \right] \right.$$

$$\left. - e(n + m)T \right] (x^n - x^{n+1}) \qquad \text{(V-87)}$$

Expanding and factoring (V-87) give

$$\mathcal{X}_m \left[e(t + T) - e(t) \right] = \lim_{k \to \infty} \left[e\left[(m + 1)T \right] x(1 - x_p) \right.$$

$$- e(mT)\, x(1 - x_p) + e\left[(m + 2)T \right] x^2(1 - x_p)$$

$$- e\left[(m + 1)T \right] x^3(1 - x_p) + \dots + e\left[(m + k + 1)T \right] x^{k+1}(1 - x_p)$$

$$\left. - e\left[(m + k)T \right] x^{k+1} (1 - x_p) \right] \qquad \text{(V-88)}$$

or

$$\frac{\mathcal{X}_m\left[e(t+T)-e(t)\right]}{(1-x_p)} = \lim_{k\to\infty}\Big[e\left[(m+1)T\right] x$$

$$- e(mT)\, x + e\left[(m+2)T\right] x^2 - e\left[(m+1)T\right] x^3 + \ldots$$

$$+ e\left[(m+k+1)T\right] x^{k+1} - e\left[(m+k)T\right] x^{k+1}\Big] \qquad \text{(V-89)}$$

It is observed that, if the limit is taken of each side of (V-89) as x approaches one with x^0 equal to one, (V-89) becomes

$$\lim_{x\to 1} \left.\frac{\mathcal{X}_m\left[e(t+T)-e(t)\right]}{1-x_p}\right|_{x^0=1} = e(\infty) - e(mT) \qquad \text{(V-90)}$$

However, from (V-86) one obtains

$$\lim_{x\to 1} \left.\frac{\mathcal{X}_m\left[e(t+T)-e(t)\right]}{1-x_p}\right|_{x^0=1}$$

$$= \lim_{x\to 1} \left. E(x, m)\right|_{x^0=1} - e(mT) \qquad \text{(V-91)}$$

Equating the right-hand sides of (V-90) and (V-91) and solving for $e(\infty)$ give

$$e(\infty) \;=\; \lim_{x \to 1} \; E(x,\, m)\Big|_{x^{0} = 1} \qquad \text{Q.E.D.} \tag{V-92}$$

An example of the use of the final value theorem for modified x-transforms is given in Example 11, Appendix B.

<u>Complex translation.</u> - If $\mathcal{X}_m\left[e(t)\right] = E(x,\, m)$, then

$$\mathcal{X}_m\left[e^{\pm at}\; e(t)\right] \;=\; (1 - x)\; e^{\pm aT(m-1)}\; E'(xe^{\pm aT},\, m) \tag{V-93}$$

where

$$E'(xe^{\pm aT},\, m) \;=\; \frac{E(xe^{\pm aT},\, m)}{1 - xe^{\pm aT}} \tag{V-94}$$

<u>Proof</u>: By definition

$$\mathcal{X}_m\left[e^{\pm at}\; e(t)\right] \;=\; x \sum_{n=0}^{\infty} e\left[(n + m)T\right]\; e^{\pm a(n+m)T}\; (x^{n} - x^{n+1})$$

$$= e^{\pm amT}\; x \sum_{n=0}^{\infty} e\left[(n + m)T\right]\; e^{\pm anT}\; (x^{n} - x^{n+1})$$

$$= e^{\pm amT}\; x\; (1 - x) \sum_{n=0}^{\infty} e\left[(n + m)T\right] \cdot (e^{\pm aT}\; x)^{n} \tag{V-95}$$

Multiplying the right-hand side of (V-94) by $e^{\pm aT} \; e^{\mp aT}$ gives

$$\mathcal{X}_m \left[e^{\pm at} \, e(t) \right]$$

$$= e^{\pm amT} \, e^{\mp aT} \, (e^{\pm aT} x)(1 - x) \sum_{n=0}^{\infty} e\left[(n + m)T\right] \, (e^{\pm aT} x)^n \tag{V-96}$$

$$= (1 - x) \, e^{\pm aT(m-1)} \, (xe^{\pm aT}) \sum_{n=0}^{\infty} e\left[(n + m)T\right] \, (e^{\pm aT} x)^n$$

$$= (1 - x) \, e^{\pm aT(m-1)} \, E'(xe^{\mp aT}, m) \tag{V-97}$$

where

$$E'(xe^{\pm aT}, m) = xe^{\pm aT} \sum_{n=0}^{\infty} e\left[(n + m)T\right] \, (xe^{\pm aT})^n$$

$$= \frac{E(xe^{\pm aT}, m)}{1 - xe^{\pm aT}} \qquad \text{Q.E.D.} \tag{V-98}$$

For an example see Example 12, Appendix B.

VI. CONCLUSIONS

The x-transform theory is based on a sampler-hold combination. This combination is quite prevalent in practical sampled-data systems. The use of the hold device is necessitated because of the higher frequencies generated by the sampler.

The x-transform theory parallels the z-transform theory and in several cases results in some simplifications. In each case the x-transform of a function of time is of no more complexity than the z-transform, and in some cases the order of the denominator of the x-transform is one less than that of the equivalent z-transform. The x-transform final value theorem is simply the sum of the coefficients of the inverse x power series. In stability studies, the Nyquist path in the x_p-plane is such that the one path, the unit circle centered at the origin, is sufficient. A z-transform Nyquist study actually requires two paths.

The z-transform is based on impulse modulation. The fact that the sampling pulse in a practical system is not an impulse causes the z-transform method to give unrealistic or sometimes even incorrect results.[18] The x-transform method, where applicable, will always give realistic results. The x-transform method is applicable to sampled-data systems in which each sampler is followed by a zero-order hold.

There are some manipulations with the term x in the x-transform which require mastering before the x-transform application provides a smooth analysis. The special rule for obtaining the x-transform of a transfer function is another possible source of error to one unfamiliar with the transform.

It is assumed that the x-transform can be extended to the area of nonlinear control systems; however, this is a subject for future study.

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APPENDIX A

TABLE 1

$\mathcal{L}$-TRANSFORMS, X-TRANSFORMS, AND MODIFIED X-TRANSFORMS

Laplace Transform	Time Function	X-Transform	Modified X-Transform
$E(s)$	$e(t)$	$E(x)$	$E(x, m)$
$\frac{1}{s}$	$u(t)$	x^0	x
$\frac{1}{s^2}$	t	$\frac{Tx}{1 - x}$	$T\left[mx + \frac{x^2}{1 - x}\right]$
$\frac{2!}{s^3}$	t^2	$\frac{T^2(x + x^2)}{(1 - x)^2}$	$T^2\left[m^2 x + \frac{(2m + 1)x^2}{1 - x} + \frac{2x^3}{(1 - x^2)}\right]$
$\frac{(n - 1)!}{s^n}$	t^{n-1}	$\lim_{a \to 0} (-1)^{n-1} \frac{d^{n-1}}{da^{n-1}}\left[\frac{1-x}{1-e^{-aT}x}\right]$	$\lim_{a \to 0} (-1)^{n-1} \frac{d^{n-1}}{da^{n-1}}\left[\frac{(x-x^2)e^{-amT}}{1-e^{-amT}x}\right]$
$\frac{1}{s+a}$	e^{-at}	$\frac{x^0 - x}{1-e^{-aT}x}$	$\frac{e^{-amT}(x-x^2)}{1-e^{-aT}x}$

Table 1 continued

$\frac{1}{(s+a)(s+b)}$	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{x^0-x}{b-a}\left[\frac{1}{1-e^{-aT}x}-\frac{1}{1-e^{-bt}x}\right]$	$\frac{1}{b-a}\left[\frac{(x-x^2)e^{-amT}}{1-e^{-aT}x}-\frac{(x-x^2)e^{-bmT}}{1-e^{-bT}x}\right]$
$\frac{1}{s(s+a)}$	$\frac{1}{a}\left[u(t)-e^{-at}\right]$	$\frac{1}{a}\left[x^0-\frac{x^0-x}{1-e^{-aT}x}\right]$	$\frac{1}{a}\left[x-\frac{(x-x^2)e^{-amT}}{1-e^{-aT}x}\right]$
$\frac{1}{s^2(s+a)}$	$\frac{1}{a}\left[t-\frac{1-e^{-at}}{a}\right]$	$\frac{1}{a}\left[\frac{Tx}{1-x}-\frac{x^0}{a}+\frac{x^0-x}{a(1-e^{-aT}x)}\right]$	$\frac{1}{a}\left[\frac{Tx^2}{1-x}+\frac{(amT-1)x}{a}+\frac{e^{-amT}(x-x^2)}{a(1-e^{-aT}x)}\right]$
$\frac{(s+b)}{s^2(s+a)}$	$\frac{a-b}{a^2}u(t)+\frac{b}{a}t$ $+\frac{1}{a}\left[\frac{b}{a}-1\right]e^{-at}$	$\frac{1}{a}\left[\frac{bTx}{1-x}+\frac{(a-b)(1-e^{-aT})x}{a(1-e^{-aT}x)}\right]$	$\frac{1}{a}\left[\frac{bTx^2}{(1-x)}+(bmT+1-\frac{b}{a})x\right.$ $\left.+\left(\frac{b-a}{a}\right)\left(\frac{e^{-amT}(x-x^2)}{1-e^{-aT}x}\right)\right]$
$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}\left[u(t)+\frac{b}{a-b}e^{-at}\right.$ $\left.-\frac{a}{a-b}e^{-bt}\right]$	$\frac{1}{ab}\left[x^0+\frac{b(x^0-x)}{(a-b)(1-e^{-aT}x)}\right.$ $\left.-\frac{a(x^0-x)}{(a-b)(1-e^{-bT}x)}\right]$	$\frac{1}{ab}\left[x+\frac{be^{-amT}x}{(a-b)(1-e^{-aT}x)}\right.$ $\left.-\frac{ae^{-bmT}x}{(a-b)(1-e^{-bT}x)}\right]$

Table 1 continued

$\frac{1}{(s+a)^2}$	te^{-at}	$\frac{(x-x^2)Te^{-aT}}{(1-e^{-aT}x)^2}$	$\frac{Te^{-amT}\left[mx + x^2(e^{-aT}-me^{-aT}-m) + x^3(me^{-aT}-e^{-aT})\right]}{(1-e^{-aT}x)^2}$
$\frac{1}{s^3(s+a)}$	$\frac{1}{2a}\left[t^2 - \frac{2}{a}t + \frac{2}{a^2}u(t) - \frac{2}{a^2}e^{-at}\right]$	$\frac{1}{a}\left[\frac{T^2x^2}{(1-x)^2} + \frac{(aT-2)Tx}{2a(1-x)} + \frac{x^0}{a^2} - \frac{x^0-x}{a^2(1-e^{-aT}x)}\right]$	$\frac{1}{a}\left\{\frac{T^2x^3}{(1-x)^2} + \frac{\left[T^2(m+\frac{1}{2})-T/a\right]x^2}{1-x} + \frac{(amT)^2x}{a^2(2-amT+1)} - \frac{e^{-amT}(x-x^2)}{a^2(1-e^{-aT}x)}\right\}$
$\frac{a}{s^2+a^2}$	sin at	$\frac{(x-x^2)\sin aT}{1-2x\cos aT + x^2}$	$\frac{(x-x^2)\sin maT + (x^2-x^3)\sin(1-m)aT}{1-2x\cos aT + x^2}$
$\frac{s}{s^2+a^2}$	cos at	$\frac{x^0-x+(x^2-x)\cos aT}{1-2x\cos aT + x^2}$	$\frac{(x-x^2)\cos maT-(x^2-x^3)\cos(1-m)aT}{1-2x\cos aT + x^2}$
$\frac{a}{s^2-a^2}$	sin h at	$\frac{(x-x^2)\sin h\, aT}{1-2x\cos h\, aT + x^2}$	$\frac{(x-x^2)\sin hamT+(x^2-x^3)\sinh(1-m)aT}{1-2x\cos haT + x^2}$

Table 1 continued

$\dfrac{s}{s^2 - a^2}$	$\cos h\ at$	$\dfrac{x^2 - x + (x^2 - x)\cos h\ aT}{1 - 2x \cos h\ aT + x^2}$	$\dfrac{(x-x^2)\cosh amT - (x^2-x^3)\cosh(1-m)aT}{1 - 2x \cos h\ aT + x^2}$
$\dfrac{a}{s(s^2 + a^2)}$	$\dfrac{1}{a}\left[u(t) - \cos at\right]$	$\dfrac{1}{a}\left[x^0 - \dfrac{x^0 - x + (x^2 - x)\cos aT}{1 - 2x\cos aT + x^2}\right]$	$\dfrac{1}{a}\left[x - \dfrac{(x-x^2)\cos amT - (x^2-x^3)\cos(1-m)aT}{1 - 2x \cos aT + x^2}\right]$
$\dfrac{a^2}{s^2(s^2 + a^2)}$	$t - \dfrac{1}{a}\sin at$	$\dfrac{Tx}{1-x} - \dfrac{1}{a}\left[\dfrac{(x-x^2)\sin aT}{1 - 2x\cos aT + x^2}\right]$	$\left[mTx + \dfrac{T}{(1-x)} - \dfrac{(x-x^2)\sin amT + (x^2-x^3)\sin(1-m)at}{a(1 - 2x\cos aT + x^2)}\right]$
$\dfrac{1}{s(s + a)^2}$	$\dfrac{1}{a^2}\left[u(t) - (1+at)e^{-at}\right]$	$\dfrac{1}{a^2}\left[x^0 - \dfrac{x^0 - x}{1 - e^{-aT}x} - \dfrac{(x-x^2)aTe^{-aT}}{(1 - e^{-aT}x)^2}\right]$	$\dfrac{1}{a^2}\left[x - \dfrac{(1-amT)(x-x^2)e^{-amT}}{1 - e^{-aT}x} + \dfrac{aTe^{-amT}(x^2-x^3)}{(1 - e^{-aT}x)^2}\right]$

Table 1 continued

$\frac{1}{s^2(s+a)^2}$	$\frac{t}{a^2} - \frac{2}{a^3}u(t) + \left(\frac{t}{a^2} + \frac{2}{a^3}\right)e^{-at}$	$\frac{1}{a^3}\left[\frac{(aT+2)x}{(1-x)} - 2x^0 + \frac{2(x^0-x)}{1-e^{-aT}x} + \frac{aTe^{-aT}(x-x^2)}{(1-e^{-aT}x)^2}\right]$	$\frac{1}{a^3}\left\{\frac{aTx^2}{(1-x)} + (amT-2)x + \left[\frac{aTe^{-aT}(x^2-x^3)}{(1-e^{-aT}x)^2} - \frac{(amT-2)(x-x^2)}{1-e^{-aT}x}\right]e^{-amT}\right\}$
$\frac{1}{(s+a)^2+b^2}$	$\frac{1}{b}e^{-at}\sin bt$	$\left(\frac{x-x^2}{b}\right)\left[\frac{e^{-aT}\sin bT}{1-2xe^{-aT}\cos bT+e^{-2aT}x^2}\right]$	$\left(\frac{1}{b}\right)\left[\frac{e^{-amT}\left[(x-x^2)\sin bmT+(x^2-x^3)\sin(1-m)bT\right]}{1-2xe^{-aT}\cos bT+e^{-2aT}x^2}\right]$
$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$	$\frac{x^0-x+(x^2-x)e^{-aT}\cos bT}{1-2xe^{-aT}\cos bT+e^{-2aT}x^2}$	$\frac{e^{-amT}\left[x\cos bmT+x^2e^{-aT}\sin(1-m)bT\right]}{1-2xe^{-aT}\cos bT+e^{-2aT}x^2}$

APPENDIX B

NUMERICAL EXAMPLES

Example 1

Determine by the partial fraction method the inverse x-transform of

$$E(x) = \frac{x(1 - e^{-aT})}{1 - e^{-aT} x_p}$$

where a is a constant and T is the sampling period in seconds. The solution is as follows:

$E(x) / (x^0 - x)$ is written as

$$\frac{E(x)}{(x^0 - x)} = \frac{x(1 - e^{-aT})}{(1 - e^{-aT} x_p)(x^0 - x)}$$

Through the use of the x-algebra, the above equation may be written as

$$\frac{E(x)}{x^0(1 - x_p)} = \frac{x^0 x_p(1 - e^{-aT})}{x^0(1 - x_p)(1 - e^{-aT} x_p)} = \frac{x_p(1 - e^{-aT})}{(1 - x_p)(1 - e^{-aT} x_p)}$$

Performing the partial fraction expansion yields

$$\frac{E(x)}{x^{0}(1 - x_p)} = \frac{1}{1 - x_p} - \frac{1}{1 - e^{-aT} x_p}$$

Solving for E(x) gives

$$E(x) = x^{0} - \frac{x^{0} - x}{1 - e^{-aT} x}$$

The inverse may now be determined through the use of the table in Appendix A as

$$\overline{e(t)} = \sum_{n=0}^{\infty} \left[u(t - nT) - u\left[t - (n + 1)T \right] \right]$$

$$- \sum_{n=0}^{\infty} e^{-anT} \left[u(t - nT) - u\left[t - (n + 1)T \right] \right]$$

<u>Example 2</u>

Determine by the use of the inversion formula the inverse x-transform of

$$E(x) = \frac{(1 - e^{-aT})\, x}{1 - e^{-aT} x_p}$$

The solution is as follows:

The inversion formula is stated as

$$e(nT)\left[u(nT) - u\left[(n-1)T\right]\right] = \frac{1}{2\pi j}\oint_{\Gamma} \frac{E(x)\ x_p^{-(n+1)}}{x^0 - x}\ dx_p$$

$$= -\sum \text{Residues of } \frac{E(x)\ x_p^{-(n+1)}}{x^0 - x}$$

$$\text{at the poles of } \frac{E(x)\ x_p^{-(n+1)}}{x^0 - x}$$

Substituting for E(x) gives

$$e(nT)\left[u(nT) - u\left[(n-1)T\right]\right]$$

$$= -\sum \text{Residues of } \frac{(1 - e^{-aT})\ x_p^{-(n+1)}\ x_p\ x^0}{(1 - e^{-aT}\ x_p)(1 - x_p)\ x^0}$$

$$\text{at the poles of } \frac{(1 - e^{-aT})\ x_p^{-n}}{(1 - e^{-aT}\ x_p)(1 - x_p)}$$

$$= 1 - e^{-anT}$$

Therefore,

$$\overline{e(t)} = \sum_{n=0}^{\infty} \left[u(t - nT)\, u\left[t - (n + 1)T\right] \right]$$

$$- \sum_{n=0}^{\infty} e^{-anT} \left[u(t - nT) - u\left[t - (n + 1)T\right] \right]$$

This agrees with the inverse transform obtained in Example 1, Appendix B.

<u>Example 3</u>

Determine by the use of the shifting theorem in Chapter II the x-transform of $e(t + T)$ if $e(t) = e^{at}$. The solution is as follows: From the table of Appendix A

$$\mathcal{X}\left[e^{-at}\right] = \frac{x^0 - x}{1 - e^{-aT}\, x}$$

The shifting theorem is given as

$$\mathcal{X}\left[e(t + T)\right] = x_p^{-1} \left[E(x) - e(0)(x^0 - x)\right]$$

and

$$\mathcal{X}\left[e^{-a(t + T)}\right] = x_p^{-1}\left[\frac{x^0 - x}{1 - e^{-aT}x} - x^0 + x\right]$$

Simplifying gives

$$\mathcal{X}\left[e^{-a(t + T)}\right] = \frac{e^{-aT}x^0 - e^{-aT}x}{1 - e^{-aT}x}$$

Let a = 1 and T = 1. Then

$$\mathcal{X}\left[e^{-(t + 1)}\right] = \frac{0.368\,x^0 - 0.368\,x}{1 - 0.368\,x}$$

$$= 0.368\,x^0 - 0.233\,x - 0.086\,x^2 - 0.0316\,x^3 - \ldots$$

The inverse of the last equation is plotted in Figure 38.

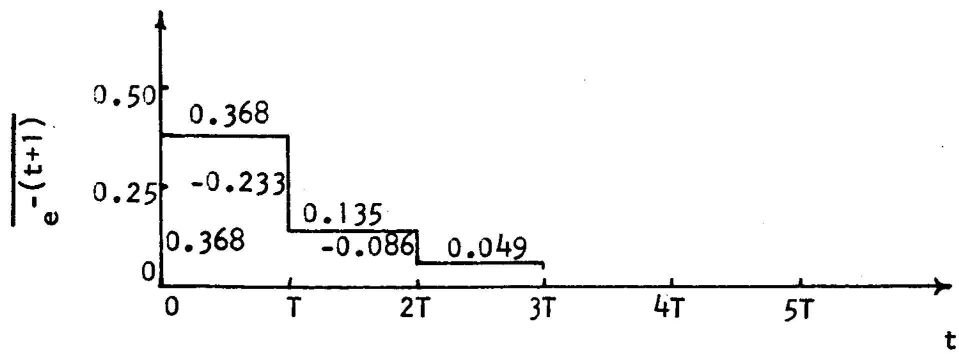

Fig. 38. - Plot of $\mathcal{X}\left[e^{-(t+1)}\right]$ illustrating the result of using the shifting theorem for a time-advance case.

Example 4

Determine by the use of the shifting theorem the x-transform of $e(t - 2T)\ u(t - 2T)$ if $e(t) = e^{-at}$. The solution is as follows: From the table of x-transforms in Appendix A

$$\mathcal{X}\left[e^{-at}\right] = \frac{x^0 - x}{1 - e^{-aT}x}$$

The shifting theorem for this case is given as

$$\mathcal{X}\left[e(t - 2T)\ u(t - 2T)\right] = x_p^2\ E(x)$$

or

$$\mathcal{X}\left[e^{-a(t - 2T)}\ u(t - 2T)\right] = \frac{x_p^2\ (x^0 - x)}{1 - e^{-aT}\ x}$$

Let a = 1 and T = 1. Then

$$\mathcal{X}\left[e^{-(t - 2)}\ u(t - 2)\right] = \frac{x^2 - x^3}{x^0 - 0.368\ x}$$

$$= x^2 - 0.632\ x^3 - 0.232\ x^4 - 0.085\ x^5 - \ldots$$

The inverse of the last equation is plotted in Figure 39.

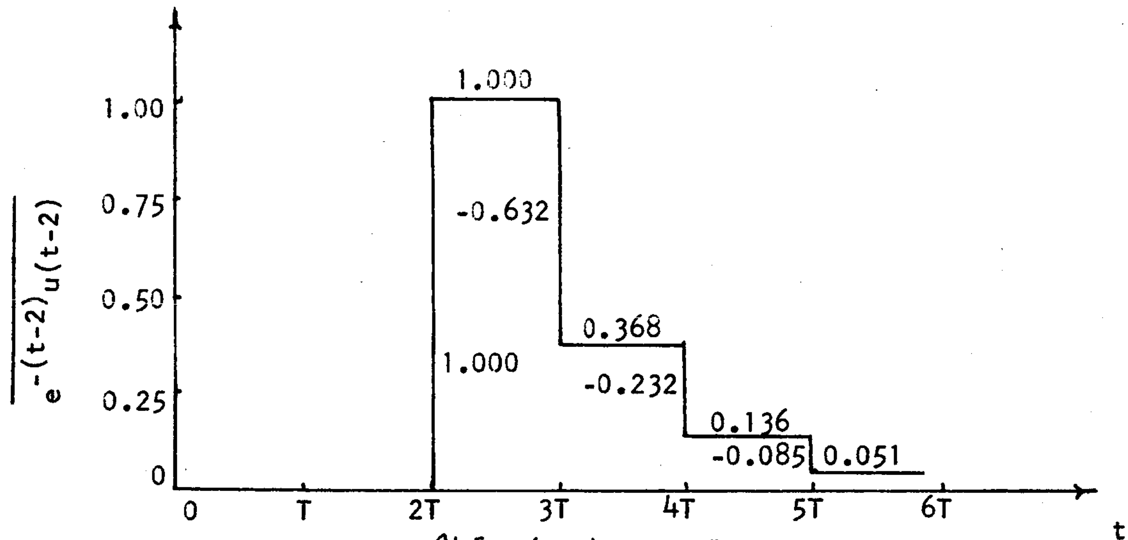

Fig. 39. - Plot of $\mathcal{Z}^{-1}\left[e^{-(t-2)}\,u(t-2)\right]$ illustrating the result of using the shifting theorem in a time-delay case.

<u>Example 5</u>

Determine the initial value of $e(t) = t$ if $E(x) = Tx \,/\, (1 - x)$.

Using the initial value theorem one obtains

$$\lim_{t \to 0} \overline{e(t)} = \lim_{x \to 0} \left[\frac{E(x)}{x^0 - x} \right]_{x^0 = 1}$$

$$= \lim_{x \to 0} \left[\frac{Tx}{(x^0 - x)^2} \right]_{x^0 = 1}$$

Therefore,

$$\lim_{t \to 0} \overline{e(t)} = 0$$

Example 6

Determine the final value of $e(t) = 1 - e^{-at}$ if $E(x) = \left[x^0 - (x^0 - x) / (1 - e^{-aT} x) \right]$. Using the final value theorem, one obtains

$$\lim_{t \to \infty} \overline{e(t)} = \lim_{x \to 1} E(x) \Big|_{x^0 = 1}$$

$$= \lim_{x \to 1} \left[\frac{x(1 - e^{-aT})}{1 - e^{-aT} x} \right]_{x^0 = 1}$$

or

$$\lim_{t \to \infty} \overline{e(t)} = 1$$

Example 7

Determine the x-transform of $t\ e^{-at}$ using complex translation. The solution is as follows:

From the x-transform table

$$\mathcal{X}\left[t\right] = \frac{Tx}{1 - x}$$

By definition

$$E'(x) = \frac{E(x)}{1 - x}$$

$$= \frac{Tx}{(1 - x)^2}$$

From the complex translation theorem

$$\mathcal{X}\left[te^{-at}\right] = (1 - x)\ E'(xe^{-aT})$$

$$= \frac{T(x - x^2)\ e^{-aT}}{(1 - e^{-aT}\ x)^2}$$

<u>Example 8</u>

Determine the modified x-transform of $e(t) = t + T$.

From the shifting theorem

$$\mathcal{X}_m\left[e(t + T)\right] = x^{-1}\ E(x,\ m) - e(mT)(x^0 - x)$$

From the table of modified x-transforms in Appendix A

$$\mathcal{X}_m\left[t\right] = Tmx + \frac{Tx^2}{1 - x}$$

Therefore

$$\mathcal{X}_m\left[t + T\right] = x^{-1}\left(Tmx + \frac{Tx^2}{1 - x}\right) - mT(x^0 - x)$$

$$= \frac{x(T + mT) - mTx^2}{1 - x}$$

Let m = 0.5 and T = 1. Then

$$\mathcal{X}_m\left[t + 1\right] = \frac{1.5\ x - 0.5\ x^2}{1 - x}$$

$$= 1.5\ x + x^2 + x^3 + x^4 + \ldots$$

A plot of $\overline{e(t + T)}$ from the last equation is given in Figure 40.

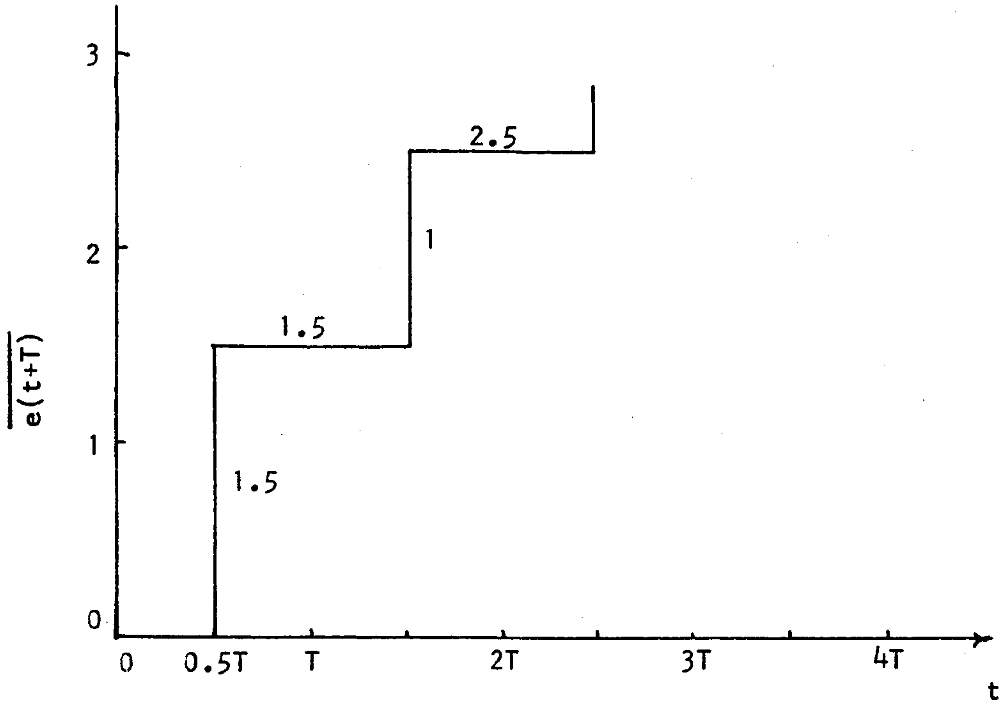

Fig. 40. - Plot of $\mathcal{X}_m \left[t + 1 \right]$ with m = 0.5.

Example 9

Determine the modified x-transform if $e(t - T)\, u(t - T) = e^{-a(t - T)}\, u(t - T)$. From the modified x-transform table

$$\mathcal{X}_m \left[e^{-at} \right] = \frac{e^{-amT}\,(x - x^2)}{1 - e^{-aT}\, x}$$

The shifting theorem for a time delay of one period gives

$$\mathcal{X}_m \left[e(t - T)\, u(t - T) \right] = x\, E(x, m)$$

and

$$\mathcal{X}_m \left[e^{-a(t - T)} u(t - T) \right] = \frac{e^{-amT} x^2 (1 - x)}{1 - e^{-aT} x}$$

Let m = 0.4, a = 1, and T = 1. Then

$$\mathcal{X}_m \left[e^{-(t-1)} u(t - 1) \right] = \frac{0.67 x^2 - 0.67 x^3}{1 - 0.368 x}$$

$$= 0.67 x^2 - 0.423 x^3 - 0.156 x^4 - 0.0574 x^5 \ldots.$$

A plot of $\overline{e(t - T)u(t - T)}$ from the last equation is given in Figure 41.

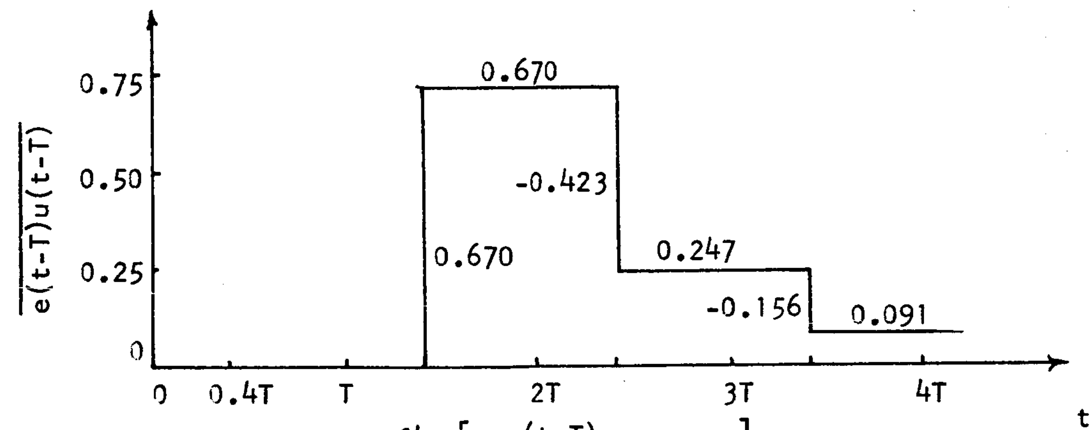

Fig. 41. - Plot of $\mathcal{X}_m \left[e^{-a(t-T)} u(t - T) \right]$ with m = 0.4, a = 1, and T = 1.

<u>Example 10</u>

Determine the initial value of e(nT, m) if

$$\mathcal{X}_m \left[e(t)\right] = \mathcal{X}_m \left[e^{-t}\right] = e^{-mT} (x - x^2) / (1 - e^{-T} x).$$

Using the initial value theorem gives

$$\lim_{t \to 0} \overline{e(t)} = \lim_{\substack{m=0 \\ x \to 0}} \left[\frac{E(x)}{x(x^0 - x)} \right]_{x^0 = 1}$$

$$= \lim_{\substack{m=0 \\ x \to 0}} \left[\frac{xe^{-mT}(x^0 - x)}{x(x^0 - x)(1 - e^{-T} x)} \right]_{x^0 = 1}$$

Through the use of the x-algebra, the last equation becomes

$$\lim_{t \to 0} \overline{e(t)} = 1$$

Example ii

Determine the final value of e(nT, m) if

$$\mathcal{X}_m \left[e(t)\right] = \mathcal{X}_m \left[\frac{1}{a} \left[u(t) - e^{-at}\right] \right]$$

$$= \frac{1}{a} \left[x - \frac{e^{-amT} (x - x^2)}{1 - e^{-aT} x} \right]$$

Using the final value theorem gives

$$\lim_{\substack{n \to \infty \\ 0 \le m \le 1}} e(nT, m) = \lim_{\substack{x \to 1 \\ 0 \le m \le 1}} E(x, m) \Big|_{x^0 = 1}$$

$$= \lim_{\substack{x \to 1 \\ 0 \le m \le 1}} \frac{1}{a} \left[x - \frac{e^{-amT} (x - x^2)}{1 - e^{-aT} x} \right]$$

$$= \lim_{\substack{x \to 1 \\ 0 \le m \le 1}} \frac{1}{a} \left[\frac{x - e^{-aT} x^2 - xe^{-amT} + x^2 e^{-amT}}{1 - e^{-aT} x} \right]$$

Under the definitions of the x-algebra, the last equation becomes

$$\lim_{\substack{n \to \infty \\ 0 \le m \le 1}} e(nT, m) = \frac{1}{a}$$

<u>Example 12</u>

Determine the modified x-transform of $e^{-at}\, t$ using complex translation. From the table of modified x-transforms

$$\mathcal{X}_m \left[t \right] = \frac{mTx - mTx^2 + Tx^2}{1 - x}$$

By definition

$$E'(x, m) = \frac{E(x, m)}{1 - x}$$

$$= \frac{mTx - mTx^2 + Tx^2}{(1 - x)^2}$$

Using the complex translation theorem, one obtains

$$\mathcal{Z}_m \left[e^{\mp at} e(t) \right] = e^{\mp aT(m-1)} (1 - x) E'(xe^{\mp aT}, m)$$

$$= \frac{Te^{-amT} (1 - x) \left[e^{-aT} x^2 + m(x - x^2 e^{-aT}) \right]}{(1 - e^{-aT} x)^2}$$